

Tokenomics: Dynamic Adoption and Valuation*

Lin William Cong[†] Ye Li[§] Neng Wang[‡]

First Draft: February 4, 2018

This Draft: October 16, 2018

Abstract

We provide a dynamic asset-pricing model of cryptocurrencies/tokens on platforms and highlight their roles on endogenous user adoption. Tokens facilitate transactions among decentralized users and allows them to capitalize future growth of promising platforms. Tokens thus can accelerate adoption, reduce user-base volatility, and improve welfare. Token price increases non-linearly in platform productivity, users' heterogeneous transaction needs, and endogenous network size. The growth of user base starts slow, becomes explosive and volatile, and eventually tapers off. Our model can be extended to discuss platform token supply, cryptocurrency competition, and pricing assets under network externality.

JEL Classification: C73, F43, E42, L86

Keywords: Bitcoin, Blockchain, Cryptocurrency, Digital Currency, ICOs, FinTech, Network Effect, Platforms, Tokens.

*The authors thank Philip Bond, Jaime Casassus, Tom Ding, Alex Frankel, Zhiguo He, Dirk Jenter, Andrew Karolyi, Yongjin Kim, Michael Sockin, Pietro Veronesi, Johan Walden, Larry Wall, Randall Wright, Yizhou Xiao, and seminar and conference participants at Ansatz Capital, Ant Financial, Atlanta Fed & GSU CEAR Conference on Financial Stability Implications of New Technology, CEPR ESSFM Gerzensee Workshop, Chicago Booth, City University of Hong Kong International Finance Conference, Emerging Trends in Entrepreneurial Finance Conference, Finance UC 14th International Conference, Georgetown University, JOIM Conference on FinTech, London Finance Theory Group Summer Conference, Norwegian School of Economics, Rome Junior Finance Conference, SEC DERA, Shanghai Forum, Stanford SITE, University of Washington Foster School of Business, and UT Dallas Finance Conference for helpful comments. Cong gratefully acknowledges Xiao Zhang for excellent research assistance and the Center for Research in Security Prices for financial support.

[†]University of Chicago Booth School of Business. E-mail: will.cong@chicagobooth.edu

[§]The Ohio State University Fisher College of Business. E-mail: li.8935@osu.edu

[‡]Columbia Business School and NBER. E-mail: neng.wang@columbia.edu

1 Introduction

Blockchain-based cryptocurrencies and tokens have taken a central stage in the FinTech world over the past few years. According to CoinMarketCap.com, the entire cryptocurrency market capitalization has grown to hundreds of billions of US dollars globally, with active trading and uses; virtually unknown a year ago, initial coin offerings (ICOs) have attracted significant attention, having raised a total US\$3.5 billion and completed more than 200 deals in 2017 alone, according to CoinSchedule. In order to draw a line between reckless speculation and financial innovation, and understand how tokens should be regulated, it is important to first understand how cryptocurrencies or tokens (henceforth generically referred to as “tokens”) derive value and the roles they play in the development of a network economy.

To this end, we develop a dynamic model of a digital economy with endogenous user adoption and native tokens that facilitate transactions and business operations. We anchor token valuation on the fundamental productivity of the (blockchain-based) network which we refer to as the “platform”, and demonstrate how tokens derive value as an exchangeable asset with limited supply that users hold to obtain transaction surplus available solely on the platform. We then derive a token pricing formula that incorporates the user-base network effect. Importantly, we highlight two roles of tokens in business development. First, the expected price appreciation makes tokens attractive to early users, allowing them to capitalize future prospect of the platform and thereby accelerating adoption. Second, because the expected price appreciation diminishes as the platform technology matures and more users adopt, the endogenous token price change moderates the volatility of user base caused by exogenous platform-productivity shocks.

Specifically, we consider a continuous-time economy with a continuum of agents who differ in their transaction needs on the platform. We broadly interpret transaction as including not only typical money transfers (e.g., on the Bitcoin blockchain) but also smart contracting (e.g., on the Ethereum blockchain). Accordingly, we model agents’ gain from blockchain transaction as a flow utility from token holdings that depends on agent-specific transaction needs, the size of the user base, and the current productivity of the platform. The exogenous “productivity” here can be broadly interpreted to reflect general usefulness of the platform, technological advances, or regulatory changes such as banning cryptocurrency trading, etc. Importantly, the flow utility from token holdings increases with the size of user base.

In our model, agents make a two-step decision on (1) whether to incur a participation cost to join the platform, and if so, (2) how many tokens to hold, which depends on both blockchain trade surplus (“transaction motive”) and the expected future token price (“investment motive”). A key insight of our model is that users’ adoption decision not only exhibits strategic complementarity through the flow utility of token holdings (the transaction motive), but also an inter-temporal complementarity via the investment motive.

We illustrate this mechanism by considering a promising platform with a positive productivity drift. The prospective growth in productivity leads agents to expect more users to join the community in future, which induces a stronger future demand for tokens and thus a current expectation of token price appreciation. The investment motive then creates a stronger demand for tokens today and greater adoption.

We characterize the Markov equilibrium with platform productivity being the state variable and derive a token pricing formula as the solution to an ordinary differential equation with boundary conditions that rule out bubbles, in line with our goal of valuing tokens based on platform fundamentals. This formula incorporates agents’ expectations of future token price change, platform productivity, user base, and user heterogeneity. Building on the analytical characterizations, we also relate our model to the existing data of cryptocurrencies and tokens, in order to discipline model parameters and provide numerical illustrations.

Our model features rich interactions between financial markets and the real economy: the financial side operates through the endogenous determination of token prices, whereas the real side manifests itself in the user adoption and surplus flows from platform translations. Tokens affect user adoption through the expected price appreciation, while user base affects token prices through influencing the users’ flow utilities and token demands. This two-way feedback naturally prompts a question: how does a platform with embedded tokens differ from one without?

To answer this question, we compare the endogenous S -curve that describes the platform adoption in our tokenized economy with those in two benchmark economies: the first-best economy (i.e., the planner’s solution) and the tokenless economy where agents use dollars as the media of exchange. Without tokens, under-adoption of promising platforms arises because a user does not internalize the positive externality from her adoption on others. Tokens, on the other hand, can improve welfare by inducing more adoption through agents’

investment motives. In contrast, when the platform productivity on average decays over time, tokens can precipitate its demise: agents forecast a smaller user base in the future and anticipating token price depreciation, shun away from adopting and holding tokens. In sum, embedding tokens on a platform front-loads the prospect of the platform and can either accelerate the adoption of a productive platform or precipitate the abandonment of an unproductive platform.

Introducing tokens can also reduce user base volatility, making it less sensitive to productivity shocks. The key driver is again the agents' investment motive. A negative productivity shock directly reduces the user's flow utility and thus lowers user adoption. However, this negative effect is mitigated by an indirect effect through the expected token price appreciation. The intuition is as follows. A lower adoption implies that more users can be brought onto the platform in the future and therefore agents expect a stronger token price appreciation. As a result, token adoption increases. Similarly, a positive productivity shock directly increases adoption by increasing the flow utility. But as the pool of potential newcomers shrinks, the expected token price appreciation declines, discouraging agents from adoption. Overall, productivity shocks, when translated to user-base fluctuations, are dampened by the endogenous dynamics of token price.

Furthermore, our model helps to explain the large cross-sectional variation in token price in the early stage of adoption, which is in line with empirical observations. Our model also shows how the volatility of platform productivity propagates into token price volatility and how endogenous adoption amplifies this propagation.

Overall, our model sheds light on the pricing of platform assets and on the roles of tokens in peer-to-peer networks that include but is not necessarily restricted to blockchains. The blockchain technology makes it possible to introduce digital currencies and tokens on various platforms to facilitate peer-to-peer transactions, but our model also applies to trusted platforms and traditional systems, such as email protocols and online social networks.¹ In sum, we offer a dynamic model and new perspectives for asset pricing on platforms with

¹In fact, many platforms have treaded this path: Linden dollar for the game *Second Life*, WoW Gold for the game *World of Warcraft*, Facebook Credits, Q-coins for Tencent's *QQ*, Amazon Coins, to name a few. Even before the heated debate on cryptocurrencies, economists and commentators were already raising questions such as, "Could a gigantic non-sovereign like Facebook someday launch a real currency to compete with the dollar, euro, yen and the like?" (Yglesias (2012)). Gans and Halaburda (2015) provide an insightful introduction on how payment systems and platforms are related.

endogenous network externality.

Related Literature. Nakamoto (2008) proposes a blockchain-based cryptocurrency for peer-to-peer transactions. Harvey (2016) surveys the emerging literature of cryptofinance. Among early studies, Biais, Bisière, Bouvard, and Casamatta (2017) and Saleh (2017) analyze the mining or minting games through Proof-of-Work and Proof-of-Stake; Easley, O’Hara, and Basu (2017), Huberman, Leshno, and Moallemi (2017), and Cong, He, and Li (2018) study miners’ compensation, organization, and market structure; Yermack (2017) and Cao, Cong, and Yang (2018) evaluate the impact of the technology on corporate governance and financial reporting; Cong and He (2018) examine informational issues in generating decentralized consensus with implications on industrial organization.

Among contemporary theories featuring token valuation in static settings, Sockin and Xiong (2018) studies tokens as indivisible membership certificates for agents to match and trade with each other; Li and Mann (2018) studies the coordination effects of staged coin offerings; Pagnotta and Buraschi (2018) studies Bitcoin pricing on exogenous user networks.

In a dynamic setting, Biais, Bisière, Bouvard, Casamatta, and Menkveld (2018) also emphasize the fundamental value of Bitcoin from transactional benefits, and study the interaction among investors, miners and hackers. We differ by studying the joint determination of user adoption and token valuation in a framework that highlights user heterogeneity, network externalities, and most importantly, inter-temporal feedback effects. Moreover, our model is applicable to platforms owned by trusted third parties, as well as (permissionless and permissioned) blockchain platforms.²

Our paper is also broadly related to the following papers. Athey, Parashkevov, Sarukkai, and Xia (2016) emphasizes the role of learning in agents’ decisions to use Bitcoin. Catalini and Gans (2018) studies developers’ pricing of tokens to fund projects. Gandal and Halaburda (2014) and Fernández-Villaverde and Sanches (2016) consider the competition among cryptocurrencies.

We do not analyze the implications of blockchain technology on general-purpose currencies and monetary policies (e.g., Balvers and McDonald (2017) and Raskin and Yermack

²For studies on the design of tokens on platforms, such as Gans and Halaburda (2015), Halaburda and Sarvary (2016), Chiu and Wong (2015), and Chiu and Koepl (2017). Our model is complementary to this line of research by allowing flexible extensions to accommodate various design features.

(2016)). Instead, we focus on the endogenous interaction between token pricing and user adoption on platforms that serve specific transaction purposes. Our study should therefore be distinguished from the monetary literature. Also, our model differs from typical asset-pricing models as ours incorporates the effects of endogenous user base and network externalities into the asset value.

We organize the remainder of the article as follows. Section 2 sets up the model. Section 3 solves the dynamic equilibrium and derives the token valuation formula. Section 4 presents the solutions for the tokenless and first-best economies. Section 5 highlights the impact of tokens on user adoption. Section 6 analyzes token price dynamics. Section 7 provides additional institutional background and Section 8 concludes. The appendix contains all the proofs, parameter choices, and extensions.

2 A Model of Tokenized Economy

Consider a continuous-time economy where a unit measure of agents conduct peer-to-peer transactions and realize trade surpluses on a platform, e.g., a blockchain. A generic good serves as the numeraire (“dollar”). We first set up and solve the model under the risk-neutral measure. In the appendix, we calibrate the model under the physical measure.³

2.1 Blockchain Technology and Agent Heterogeneity

Platform transaction surplus. The blockchain platform allows agents to conduct peer-to-peer transactions. These transactions are settled via a medium of exchange, which can either be the numeraire (e.g., dollar) or the native token for this blockchain. We use $x_{i,t}$ to denote the value of agent i 's holdings in the unit of the numeraire. These holdings facilitate transactions on the platform and generate a flow of utility over dt given by

$$x_{i,t}^{1-\alpha} (N_t A_t e^{u_i})^\alpha dt, \tag{1}$$

³The typical no-arbitrage condition implies a probability measure—the risk-neutral measure—under which agents discount future cash flows using the risk-free rate.

where N_t is the platform user base, A_t measures blockchain productivity, u_i captures agent i 's specific needs for blockchain transactions, and $\alpha \in (0, 1)$ is a constant.⁴

The platform productivity, A_t , evolves according to a geometric Brownian motion:

$$\frac{dA_t}{A_t} = \mu^A dt + \sigma^A dZ_t^A. \quad (2)$$

We focus on the case of a promising yet risky platform, i.e., $\mu^A > 0$ and $\sigma^A > 0$. We interpret A_t broadly. A positive shock to A_t reflects technological advances in cryptography and computation, favorable regulatory or policy changes, growing users' interests, and increasing variety of activities feasible on the platform.

The transaction surplus depends on N_t , the total measure of agents on the platform (i.e., $x_{i,t} > 0$). This specification captures the network externality among users, such as the greater ease of finding trading or contracting counterparties in a larger community.

User heterogeneity and adoption. We assume that agents' transaction needs, u_i , are heterogeneous. Let $G(u)$ and $g(u)$ denote the cross-sectional cumulative distribution function and the density function of u_i that is assumed to be continuously differentiable over a finite support $[\underline{U}, \bar{U}]$.

u_i can be broadly interpreted. For payment blockchains (e.g., Ripple and Bitcoin), a high value of u_i reflects agent i 's urge to conduct a transaction (e.g., an international remittance and a purchase of drugs). For smart-contracting blockchains (e.g., Ethereum), u_i captures agent i 's project productivity. For decentralized computation (e.g., Dfinity) and data storage (e.g., Filecoin) applications, u_i corresponds to the need for secure and fast access to computing power and data.

To join the platform and realize the transaction surplus, an agent incurs a flow cost ϕdt . For example, transacting on the platform takes effort and attention. At any time t , agents may choose not to participate and then collect no transaction surplus. Therefore, agents with sufficiently high u_i choose to join the platform, while agents with sufficiently low u_i do not participate.

⁴Our results are qualitatively robust to alternative specifications that feature decreasing total return i.e., $(x_{i,t})^{1-\alpha-\gamma} (N_t A_t e^{u_i})^\alpha$ with $\gamma > 0$.

2.2 Tokens, Agents' Problem, and Equilibrium

Tokens and endogenous price. In what follows, we focus on the joint dynamics of token valuation and user adoption on platforms requiring native tokens as the medium of exchange, i.e.,

$$x_{i,t} = P_t k_{i,t}, \quad (3)$$

where P_t is the unit price of token in terms of the numeraire and $k_{i,t}$ is the units of token.⁵

We conjecture and verify that the equilibrium price dynamics is a diffusion process,

$$dP_t = P_t \mu_t^P dt + P_t \sigma_t^P dZ_t^A, \quad (4)$$

where μ_t^P and σ_t^P are endogenously determined.

Throughout the paper, we use upper-case letters for aggregate and price variables that individuals take as given, and lower-case letters for individual-level variables.

Agent's problem. Let $y_{i,t}$ denote agent i 's *cumulative* profit from blockchain activities. Agent i then maximizes life-time utility under the risk-neutral measure,

$$\mathbb{E} \left[\int_0^\infty e^{-rt} dy_{i,t} \right], \quad (5)$$

where we can write the incremental profit $dy_{i,t}$ as follows:

$$dy_{i,t} = \max \left\{ 0, \max_{k_{i,t} > 0} \left[(P_t k_{i,t})^{1-\alpha} (N_t A_t e^{u_i})^\alpha dt + k_{i,t} \mathbb{E}_t [dP_t] - \phi dt - P_t k_{i,t} r dt \right] \right\}. \quad (6)$$

Here, the outer “max” operator reflects agent i 's option to leave the platform and obtain zero profit, and the inner “max” operator reflects agent i 's optimal choice of $k_{i,t}$.

Inside the inner max operator are four terms that add up to give the incremental profits from participating in the blockchain transaction. The first term corresponds to the blockchain trade surplus given in (1). The second term is the expected capital gains from

⁵The dollar value of inputs ($P_t k_{i,t}$), instead of $k_{i,t}$ alone, shows up in the surplus flow to facilitate the comparison between platforms with and without tokens. It is also motivated by the fact that the economic value of blockchain trades depends on the numeraire value of real goods and services that are transacted in tokens. Our results are qualitatively similar in the alternative specification with only $k_{i,t}$ (instead of $P_t k_{i,t}$) in the trade surplus.

holding $k_{i,t}$ units of tokens, where $\mathbb{E}_t [dP_t] = P_t \mu_t^P dt$. Users care about the sum of the on-chain transaction surplus and the expected token appreciation given by the first two terms in (6). The third term is the participation cost and the last term is the financing cost of holding $k_{i,t}$ units of tokens.

It is worth emphasizing that in our tokenized economy, agents must hold tokens for at least an instant dt to complete transactions and derive utility flows. This holding period exposes users to token price change over dt , but is important for blockchain transactions for several reasons. One example is smart contracting, which often requires tokens as collateral and therefore exposes the collateral owners to token price fluctuations. We provide more examples and institutional details that motivate our setting in Section 7.

The Markov equilibrium. We study a Markov equilibrium with A_t , the only source of exogenous shocks in the economy, as the state variable. For simplicity, we fix the token supply to a constant M .⁶ The market clearing condition is

$$M = \int_{i \in [0,1]} k_{i,t} di, \tag{7}$$

where for those who do not participate, $k_{i,t} = 0$.

Definition 1. *A Markov equilibrium with state variable A_t is described by agents' decisions and equilibrium token price such that the token market clearing condition given by Equation (7) holds and agents optimally decide to participate (or not) and choose token holdings.*

3 Dynamic Equilibrium of Adoption and Valuation

We now solve for the Markov equilibrium, where the user base, N_t , users' token holdings, $k_{i,t}$, and token price P_t , are functions of the state variable A_t . First, we analyze agents' decision to participate and hold tokens, given A_t and agents' expectation of token price change μ_t^P . Then we complete the solution by solving the token price dynamics (and in particular, μ_t^P as a function of A_t). Each step ends with a summarizing proposition.

⁶This is the case with many ICOs that fix the supply of tokens. More generally, the blockchain technology allows supply schedules to be based on explicit rules independent of any endogenous variables, and can be accommodated in the model by adding in the inflation or deflation of token prices corresponding to the rules.

Token demand and user base. Conditioning on joining the platform, agent i chooses the optimal token holdings, $k_{i,t}^*$, by using the first order condition,

$$(1 - \alpha) \left(\frac{N_t A_t e^{u_i}}{P_t k_{i,t}^*} \right)^\alpha + \mu_t^P = r, \quad (8)$$

which states that the sum of *marginal* transaction surplus on the platform and the expected token price change is equal to the required rate of return, r . Rearranging this equation, we obtain the following expression for the optimal token holdings:

$$k_{i,t}^* = \frac{N_t A_t e^{u_i}}{P_t} \left(\frac{1 - \alpha}{r - \mu_t^P} \right)^{\frac{1}{\alpha}}. \quad (9)$$

$k_{i,t}^*$ has several properties. First, agents hold more tokens when the common productivity, A_t , or agent-specific transaction need, u_i , is high, and also when the user base, N_t , is larger because it is easier to conduct trades on the platform. Equation (9) reflects an investment motive to hold tokens, that is $k_{i,t}^*$ increases in the expected token appreciation, μ_t^P .

Using $k_{i,t}^*$, we obtain the following expression for the agent's profit conditional on participating on the platform:

$$N_t A_t e^{u_i} \alpha \left(\frac{1 - \alpha}{r - \mu_t^P} \right)^{\frac{1-\alpha}{\alpha}} - \phi. \quad (10)$$

Agent i will only participate when the preceding expression is non-negative. That is, only those agents with sufficiently large u_i will participate. Let \underline{u}_t denote the type of user who is indifferent between participating on the platform or not.

By setting the expression (10) to zero, we obtain the following equation for the user-type cutoff threshold:

$$\underline{u}_t = \underline{u}(N_t; A_t, \mu_t^P) = -\ln(N_t) + \ln\left(\frac{\phi}{A_t \alpha}\right) - \left(\frac{1 - \alpha}{\alpha}\right) \ln\left(\frac{1 - \alpha}{r - \mu_t^P}\right). \quad (11)$$

Because only agents with $u_i \geq \underline{u}_t$ will participate, the user base is then given by

$$N_t = 1 - G(\underline{u}_t). \quad (12)$$

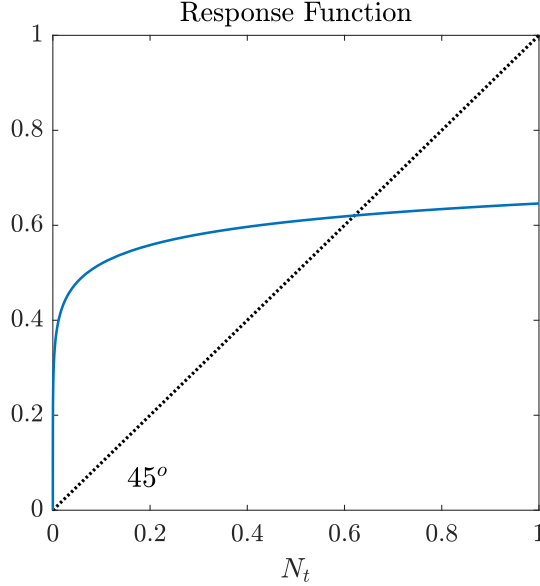


Figure 1: **Determining User Base.** This graph shows the aggregate response of users' adoption decision, $R(n; A_t, \mu_t^P)$, to different levels of $N_t = n \in [0, 1]$, given A_t and μ_t^P .

The adoption threshold \underline{u}_t is decreasing in A_t because a more productive platform attracts more users. The threshold also decreases when agents expect a higher token price appreciation (i.e., higher μ_t^P).

Equations (11) and (12) jointly determine the user base N_t given A_t and μ_t^P . First, we note that zero adoption is always a solution. Next, we focus on the non-degenerate case, i.e., $N_t > 0$. Fixing A_t and μ_t^P , we consider a *response* function $R(n; A_t, \mu_t^P)$ that maps a hypothetical value of N_t , say n , to the measure of agents who choose to participate after knowing $N_t = n$. As depicted in Figure 1, the response curve originates from zero (the degenerate case). In the Appendix, we first show that given μ_t^P , there exists a threshold $\underline{A}(\mu_t^P)$ such that for $A_t < \underline{A}(\mu_t^P)$, a non-degenerate solution does not exist, because the response curve never crosses the 45° line. Then we prove that when $A_t \geq \underline{A}(\mu_t^P)$, the response curve crosses the 45° line exactly once (and from above) under the assumption that the hazard rate for $g(u)$ is increasing.⁷

Proposition 1 (Token Demand and User Base). *Given μ_t^P and a sufficiently high productivity, i.e., $A_t > \underline{A}(\mu_t^P)$, we have a unique non-degenerate solution, N_t , for Equations*

⁷The hazard rate, $\frac{g(u)}{1-G(u)}$, is increasing in u if and only if $1 - G(u)$ is log-concave. This assumption is common in the theory literature to avoid the technically complicated “ironing” of virtual values.

(11) and (12) under the increasing hazard-rate assumption. The user base, N_t , increases in μ_t^P and A_t . Agent i participates when $u_i \geq \underline{u}_t$, where \underline{u}_t is given by Equation (11). Conditional on participating, Agent i 's optimal token holding, $k_{i,t}^*$, is given by Equation (9). The token holding, $k_{i,t}^*$, decreases in P_t and increases in A_t , μ_t^P , u_i , and N_t .

Token Pricing. First, we define the *participants' aggregate transaction need* as

$$S_t := \int_{\underline{u}_t}^{\bar{U}} e^u g(u) du, \quad (13)$$

the integral of e^{u_i} of participating agents. Substituting optimal holdings in Equation (9) into the market clearing condition in Equation (7), we obtain the *Token Pricing Formula*:

$$P_t = \frac{N_t S_t A_t}{M} \left(\frac{1 - \alpha}{r - \mu_t^P} \right)^{\frac{1}{\alpha}}. \quad (14)$$

The token price increases in N_t – the larger the user base is, the higher trade surplus individual participants can realize by holding tokens, and stronger the token demand. The price-to-user base ratio increases in the platform productivity, the expected price appreciation, and the network participants' aggregate transaction need, while it decreases in the token supply M . The formula reflects certain observations by practitioners, such as incorporating DAA (daily active addresses) and NVT Ratio (market cap to daily transaction volume) in token valuation framework, but instead of heuristically aggregating such inputs into a pricing formula, we solve *both* token pricing and user adoption as an equilibrium outcome.⁸

The token pricing formula given in Equation (14) also implies the following differential equation that characterizes $P(A_t)$ as a function of state variable A_t .⁹

$$\mu^A A_t \left(\frac{dP_t}{dA_t} \right) + \frac{1}{2} (\sigma^A)^2 A_t^2 \frac{d^2 P_t}{dA_t^2} + (1 - \alpha) \left(\frac{N_t S_t A_t}{M P_t} \right)^\alpha P_t - r P_t = 0. \quad (15)$$

⁸See, for example, *Today's Crypto Asset Valuation Frameworks* by Ashley Lannquist at Blockchain at Berkeley and Haas FinTech.

⁹Since \underline{u}_t decreases in μ_t^P , the RHS of Equation (14) increases in μ_t^P , so A_t and P_t uniquely pin down μ_t^P , which contains the first and second derivatives of P_t to A_t by Itô's lemma. Therefore, the mapping from A_t , $P(A_t)$, and $P'(A_t)$ to $P''(A_t)$ is unique. The implied ODE satisfies all the conditions in Theorems 4.17 and 4.18 in Jackson (1968), which guarantee the existence and uniqueness of the solution. The existence of a unique (fundamental-driven) equilibrium distinguishes our paper from studies such as Sockin and Xiong (2018) that focus on equilibrium multiplicity and allows us to highlight dynamics of adoption and valuation.

We solve the preceding ODE for $P(A_t)$ with the following boundary conditions. The first is

$$\lim_{A_t \rightarrow 0} P(A_t) = 0, \quad (16)$$

which means that the token price is zero when the platform is permanently unproductive (Note that $A_t = 0$ is an absorbing state.)

Next, we discuss remaining boundary conditions by using the solution under full adoption. As $N_t = 1$, the aggregate transaction demand, S_t , is equal to \bar{S} , where

$$\bar{S} \equiv \int_{\underline{U}}^{\bar{U}} e^u g(u) du, \quad (17)$$

is the sum (integral) of all agents' e^{u_i} . Let $\bar{P}(A_t)$ denote token price under full adoption. As we focus on the fundamentals, the token price dynamics is fully determined by the underlying productivity growth, i.e., $\mu_t^P = \mu^A$. Therefore, we obtain the following Gordon Growth Formula for token price under full adoption:

$$\bar{P}(A_t) = \frac{\bar{S}A_t}{M} \left(\frac{1 - \alpha}{r - \mu^A} \right)^{\frac{1}{\alpha}}, \quad (18)$$

where \bar{S} is given by Equation (17). Let \tilde{A} denote the lowest value of A_t that induces full adoption. The value-matching and smooth-pasting conditions hold at \tilde{A} :

$$P(\tilde{A}) = \bar{P}(\tilde{A}) \quad \text{and} \quad P'(\tilde{A}) = \bar{P}'(\tilde{A}). \quad (19)$$

Proposition 2 (Markov Equilibrium). *With A_t being the state variable, the token price $P(A_t)$ uniquely solves the ODE given by Equation (14) subject to boundary conditions given by Equations (16) and (19). Given the token price dynamics, agents' optimal token holdings and participation decisions together with the user base are reported in Proposition 1.*

Figure 2 summarizes the key economic mechanism discussed thus far, where the blue dotted, black solid, and red dashed arrows show respectively the user-base externality, the transaction motive of token holdings, and the investment motive of token holdings.

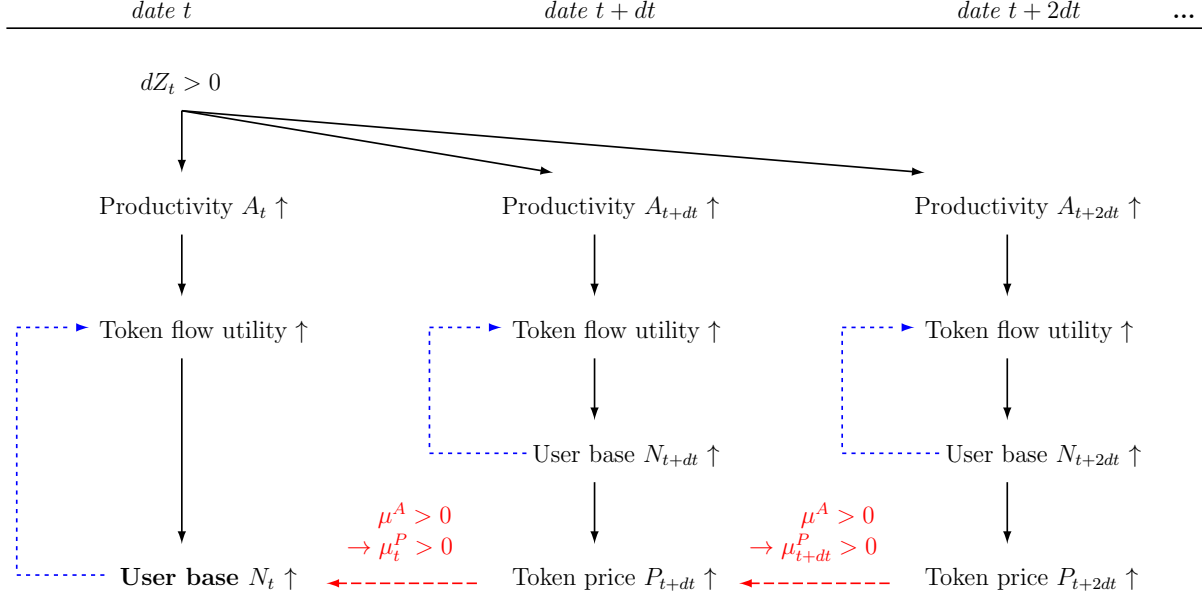


Figure 2: **The Economic Mechanism.** The black solid arrows point to the increases of the current and future (expected) levels of productivity A , which lead to higher flow utilities of tokens, and in turn, larger user bases N . The blue dotted arrows show that increases in user base result in even higher flow utility due to the contemporaneous user-base externality. Finally, more users push up the token prices P in future dates, which feed into a current expectation of price appreciation and greater adoption (red dashed arrows).

4 Benchmark Economies

This section analyzes two benchmark economies to help us understand the roles of tokens. The first is the tokenless economy, which features a platform where the medium of exchange is dollar, the numeraire. By comparing our tokenized economy with this benchmark, we highlight how introducing native tokens affects user adoption. The second benchmark is the central planner’s problem, which achieves the first-best outcome and helps us understand the welfare consequences of introducing tokens.

4.1 Tokenless Economy

Unlike the tokenized economy, in our tokenless economy, dollar, the numeraire, is the medium of exchange and agents only have transactional motives. The agent’s profit is given by

$$dy_{i,t} = \max \left\{ 0, \max_{x_{i,t} > 0} \left[\underbrace{(x_{i,t})^{1-\alpha} (N_t A_t e^{u_i})^\alpha}_{\text{blockchain trade surplus}} dt - \underbrace{\phi dt}_{\text{participation cost}} - \underbrace{x_{i,t} r dt}_{\text{financing cost}} \right] \right\}. \quad (20)$$

Unlike Equation (6) for the tokenized economy, as there is no native token, the token price fluctuation, μ_t^P , no longer appears in the agent's profits.

Conditional on joining the platform (i.e., $x_{i,t} > 0$), the agent chooses $x_{i,t}$ as follows:

$$x_{i,t}^* = N_t A_t e^{u_i} \left(\frac{1-\alpha}{r} \right)^{\frac{1}{\alpha}}. \quad (21)$$

The maximized profit when joining the platform is then

$$N_t A_t e^{u_i} \alpha \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha}{\alpha}} - \phi. \quad (22)$$

An agent joins the platform only when Expression (22) is positive. That is, agent i participates if and only if $u_i \geq \underline{u}_t^{NT}$, where \underline{u}_t^{NT} is the endogenous threshold given by

$$\underline{u}_t^{NT} = -\ln(N_t) + \ln\left(\frac{\phi}{A_t \alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right) \ln\left(\frac{1-\alpha}{r}\right). \quad (23)$$

Here, the superscript “ NT ” refers to the “no-token” case. The user base is thus given by

$$N_t^{NT} = 1 - G(\underline{u}_t^{NT}). \quad (24)$$

Equations (23) and (24) jointly determine \underline{u}_t^{NT} and N_t^{NT} as functions of A_t . Additionally, the user base, N_t^{NT} , increases in A_t .

We define \underline{A}^{NT} by following essentially the same reasoning as the one in Proposition 1 for the tokenized economy.¹⁰ We can show that there exists a non-degenerate solution, N_t^{NT} , when the platform is sufficiently productive, i.e., $A_t \geq \underline{A}^{NT}$.

Next, we consider the social planner's problem by internalizing network externalities.

4.2 The First-best (FB) Economy

Given a user base N_t , the socially optimal holdings of dollars is still

$$x_{i,t}^* = N_t A_t e^{u_i} \left(\frac{1-\alpha}{r} \right)^{\frac{1}{\alpha}}. \quad (25)$$

¹⁰We can simply solve \underline{A}^{NT} by imposing $\mu_t^P = 0$ in Proposition 1.

Let \mathcal{U}_t denote the set of participating users and in equilibrium the mass is equal to N_t . The total trade surplus (if positive) is given by

$$\int_{i \in \mathcal{U}_t} \left[\alpha N_t A_t e^{u_i} \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha}{\alpha}} - \phi \right] di = N_t \left[\alpha \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha}{\alpha}} A_t \int_{i \in \mathcal{U}_t} e^{u_i} di - \phi \right]. \quad (26)$$

To maximize this welfare flow, the planner optimally sets $N_t = 1$, i.e., \mathcal{U}_t being the full set of agents, unless given $N_t = 1$, the objective (26) is negative, in which case it is socially optimal to have zero adoption and zero welfare. The switching from zero adoption to full adoption happens at

$$\underline{A}^{FB} = \phi \left[\alpha \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha}{\alpha}} \bar{S} \right]^{-1}, \quad (27)$$

where \bar{S} is given by Equation (17). Given that $\bar{S} < \infty$, welfare maximization has a bang-bang solution, requiring full adoption if $A \geq \underline{A}^{FB}$ and zero adoption otherwise.

We show that there are more agents participating on the platform in the FB economy than in our tokenless economy, which means the productivity thresholds for adoption in the two economies satisfies $\underline{A}^{FB} < \underline{A}^{NT}$ (proof in the appendix). This result follows from that the social planner internalizes the positive externality of an agent's adoption on other users.

How does the decentralized equilibrium of tokenized economy differ from the planner's solution? On the one hand, tokens induce an investment motive in agents' decision to participate, alleviating the under-adoption problem in the decentralized tokenless equilibrium. On the other hand, over-adoption may happen in the sense that $N_t > 0$ even when $A_t < \underline{A}^{FB}$.

In the appendix, we use data on token pricing and adoption to discipline our choices of parameter values for numerical solutions. Next, we combine the analytical and numerical findings to discuss the roles of tokens.

5 The Roles of Tokens

In this section, we analyze the adoption dynamics and highlight the roles of tokens by comparing the tokenized economy, the tokenless economy, and the first-best economy.

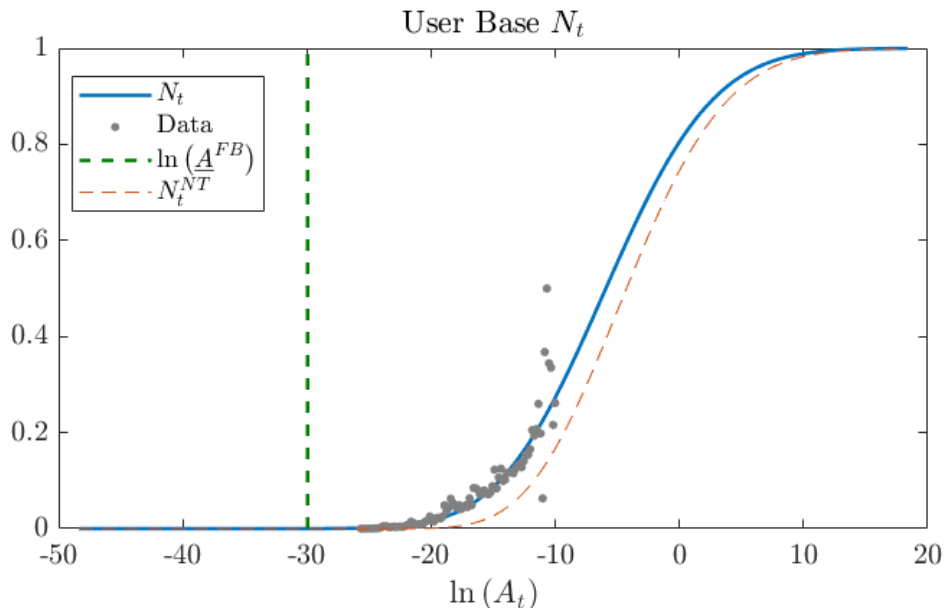


Figure 3: **Dependence of User Base on Blockchain Productivity.** This graph shows N_t , the user base of tokenized economy (blue solid curve), data of normalized active user addresses (gray scattered dots), and the user base of tokenless economy against $\ln(A_t)$, the blockchain productivity. The dotted vertical line marks the level of productivity, beyond which the planner chooses full adoption, and below which the planner chooses zero adoption.

5.1 User-Adoption Acceleration

We illustrate the adoption acceleration and user-base volatility reduction effects of tokens with the numerical solutions. When tokens are introduced as the platform’s medium of exchange, token prices reflect agents’ expectations of future technological progress and user adoption. Tokens therefore accelerate adoption because agents joining the community enjoy not only the trade surplus but also the investment return from token price appreciation.

The solid line in Figure 3 shows that the user base N_t is an S -shaped function of $\ln(A_t)$.¹¹ When the platform’s productivity A_t is low, the user base N_t barely responds to changes in A_t . In contrast, when A_t is moderately high, N_t responds much more to changes in A_t . The growth of user base feeds on itself – the more agents join the ecosystem, the higher transaction surplus each derives. User adoption eventually slows down when the pool of newcomers gets exhausted. We also plot the scattered data points. We provide details on sample construction in Appendix B.

¹¹The curve starts at $\ln(A_t) = -48.35$ ($A_t = 1e - 21$), a number that we choose to be close to zero, the left boundary. The curve ends at $\ln(A_t) = 18.42$ ($A_t = 1e8$), the touching point between $P(A_t)$ and $\bar{P}(A_t)$.

Figure 3 also compares the user adoption in tokenized and tokenless (decentralized) economies. The former strictly dominates the latter. Both economies reach full adoption when A_t becomes sufficiently large. Notice that the adoption thresholds in Equations (11) and (23) differ by the extra μ_t^P term. When $\mu_t^P > 0$, $N_t > N_t^{NT}$ given A_t , where N_t is determined by Equation (12). In other words, the expected token price appreciation induces a higher level of adoption than the case without tokens.¹²

Token price appreciation critically depends on the growth of A_t . When $\mu_A > 0$, agents forecast a higher token price and the investment motive accelerates user adoption. Without tokens, this investment-driven demand is shut down. Therefore, introducing tokens help capitalize future productivity growth and grow promising platforms.¹³ In contrast, when $\mu_A < 0$, the expected token depreciation ($\mu_t^P < 0$) precipitates user exits and the demise of the platform. Our numerical analysis focuses on the case where $\mu^A > 0$.

For comparison, we also plot in Figure 3 the first-best solution via the dotted vertical line at $\ln(\underline{A}^{FB})$, which is given by Equation (27). Recall that the planner chooses full adoption if $A_t \geq \underline{A}^{FB}$ and zero adoption otherwise. Relative to the first-best economy, a tokenless economy features under-adoption and introducing tokens helps mitigate this inefficiency.¹⁴

5.2 User-base Volatility Reduction

Next, we compare the user base volatility in tokenized and tokenless economies. Note that in the first-best economy, because the adoption is either zero or full, user base volatility is not an issue.

To derive the dynamics of N_t , we first conjecture the following equilibrium diffusion process:

$$dN_t = \mu_t^N dt + \sigma_t^N dZ_t^A. \quad (28)$$

¹²Note that in the system without token, transactions are settled on dollars, and we simplify the analysis by assuming that the price of dollar in goods is fixed at one. In reality, the value of dollar declines over time due to inflation, which strengthens the adoption acceleration effect.

¹³We note that a predetermined token supply schedule is important. If token supply can arbitrarily increase ex post, then the expected token price appreciation is delinked from the technological progress. Pre-determinacy or commitment can only be credibly achieved through the decentralized consensus mechanism empowered by the blockchain technology. In contrast, traditional monetary policy has commitment problem – monetary authority cannot commit not to supply more money when its currency value is relatively high.

¹⁴Tokenized economy may also lead to *over*-adoption because it is possible that $N_t > 0$ even when $A_t < \underline{A}^{FB}$. For most model parameter choices, over-adoption is not a severe problem because N_t is extremely close to zero for $A_t < \underline{A}^{FB}$.

In the appendix, we show that in the tokenless economy

$$\sigma_t^N = \left(\frac{g(\underline{u}_t^{NT})}{1 - g(\underline{u}_t^{NT})/N_t^{NT}} \right) \sigma^A \quad (29)$$

and in the tokenized economy

$$\sigma_t^N = \left(\frac{g(\underline{u}_t)}{1 - g(\underline{u}_t)/N_t} \right) \left[\sigma^A + \left(\frac{1 - \alpha}{\alpha} \right) \left(\frac{\sigma_t^{\mu^P}}{r - \mu_t^P} \right) \right], \quad (30)$$

where $\sigma_t^{\mu^P}$ is the diffusion of μ_t^P as defined below:

$$d\mu_t^P = \mu_t^P dt + \sigma_t^{\mu^P} dZ_t^A. \quad (31)$$

We know that N_t follows a reflected (or “regulated”) diffusion process bounded in $[0, 1]$.

Comparing Equations (29) and (30), we see that introducing tokens alters the user-base volatility through $\sigma_t^{\mu^P}$, which is the volatility of expected token appreciation, μ_t^P , as defined in Equation (31). Embedding a native token may either amplify or dampen the shock effect on the user base, depending on the sign of $\sigma_t^{\mu^P}$. By Itô’s lemma, $\sigma_t^{\mu^P} = \frac{d\mu_t^P}{dA_t} \sigma^A A_t$, so the sign of $\sigma_t^{\mu^P}$ depends on whether μ_t^P increases or decreases in A_t .

Intuitively, μ_t^P decreases in A_t (and thus, $\sigma_t^{\mu^P} < 0$), precisely because of the endogenous user adoption. Consider a positive shock to A_t , which has a direct effect of increasing N_t due to higher transaction surplus. For a large N_t , the potential for N_t to grow decreases, so does the expected token appreciation, i.e., μ_t^P . The token price dynamics therefore moderate N_t ’s increase. Similarly, consider a negative shock to A_t . The token price channel mitigates the decrease in N_t . Overall, introducing token can *reduce* the user-base sensitivity to shocks.

Next, we illustrate in Figure 4 how tokens reduce user-base volatility. The left panel plots σ_t^N , and compares the cases with and without token across different stages of adoption. Both curves start and end at zero, consistent with the S-shaped development in Figure 3 in which both curves start flat and end flat. This volatility reduction effect is more prominent in the early stage of development when A_t and N_t are low. Note that σ_t^N can be slightly higher when token is introduced because the first brackets in Equations (29) and (30) differ due to the difference between \underline{u}_t^{NT} and \underline{u}_t even for the same adoption level.

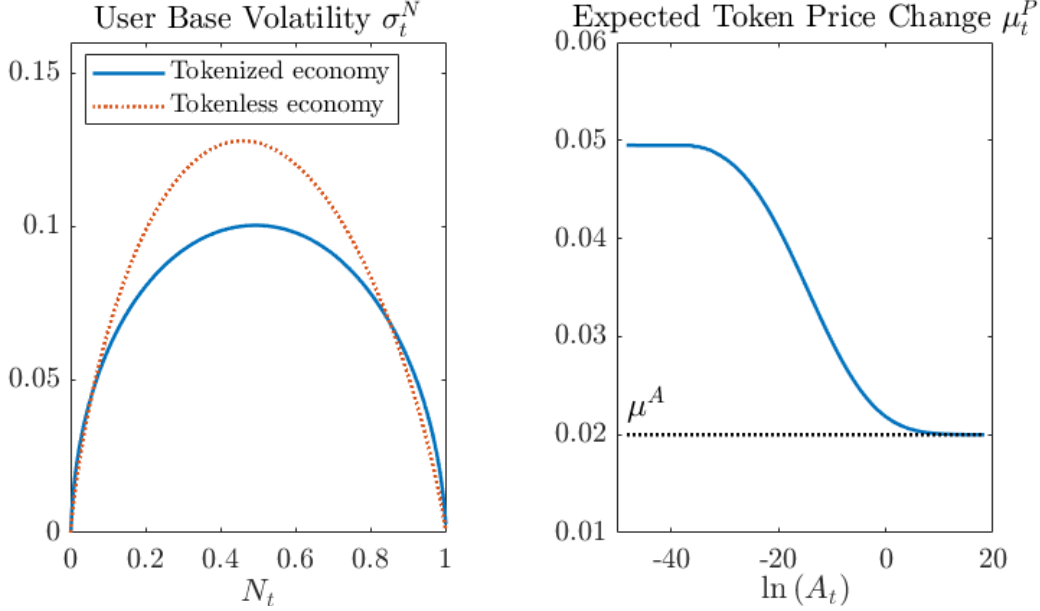


Figure 4: **User-Base Volatility Reduction Effect.** The left panel of this graph shows the volatility of user base, σ_t^N , in the tokenized (blue solid curve) and tokenless (red dotted curve) economies over adoption stages, N_t . The right panel shows the expected token price change, μ_t^P , across different levels of blockchain productivity, $\ln(A_t)$. The black dotted line marks the expected growth rate of blockchain productivity.

The right panel of Figure 4 plots μ_t^P against $\ln(A_t)$, showing their negative relation that causes $\sigma_t^{\mu^P} < 0$, which generates the volatility reduction effect. When A_t is low and N_t is low, token price is expected to increase fast, reflecting both the future growth of A_t and N_t . As A_t and N_t grow, the pool of agents who have not adopted ($1 - N_t$) shrinks and there is less potential for N_t to grow. As a result, the expected token appreciation declines.

Remark: Given the roles of the tokens in accelerating and stabilizing user adoption, entrepreneurs may want to introduce them in a platform. For example, suppose the platform can collect a fee from the users, greater adoption would increase the revenue of the platform. Tokens are often used to raise capital from early investors (e.g., in ICOs) for platform development. Through retaining some tokens, early investors and entrepreneurs also benefit from token price appreciation. Our on-going work explores such considerations of the entrepreneurs and platform designers.¹⁵

¹⁵One promising extension is the comparison of the adoption acceleration benefit of tokens we highlight with traditional user subsidy through VC capital. Bakos and Halaburda (2018) recently study this tradeoff in a two-period model without technological uncertainty or user heterogeneity.

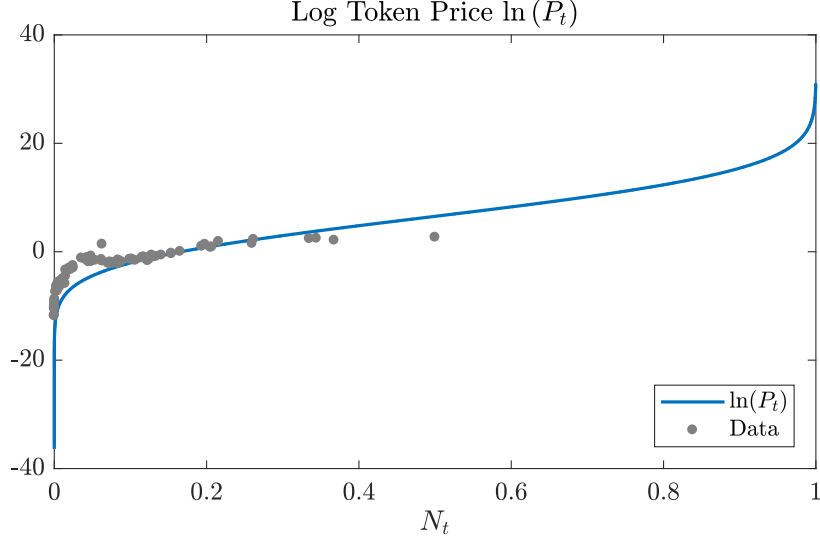


Figure 5: **Token Price Dynamics over Adoption Stages.** This graph shows the log token price across adoption stages, N_t (blue solid curve), and data as scattered dots.

6 Token Price Dynamics under Endogenous Adoption

In this section, we discuss how endogenous user adoption leads to nonlinear price dynamics that are broadly consistent with empirical observations.

Token price over adoption stages. As some key inputs, e.g., $\ln(A_t)$, in our model are unobservable, in Figure 5 we link two key observables, the logarithmic token price $\ln(P_t)$ and the user base N_t —both are functions of A_t in equilibrium. Token price increases fast with adoption in the early stage, changes more gradually in the intermediate stage, and speeds up again once the user base reaches a sufficiently high level. The two price run-ups in the initial and final stages of adoption correspond to the slow user base growth in these two stages.

This figure helps us understand the cross-sectional differences in token pricing. For exposition, we sort blockchain platforms into three adoption stages: early, intermediate, and late. For two blockchain platforms in the early stage, a small difference of N_t between them can generate a very large differences in $\ln(P_t)$. Similar result holds in the late stage. In contrast, in the intermediate stage, even a large difference of N_t between the two platforms only yields a small difference of $\ln(P_t)$.

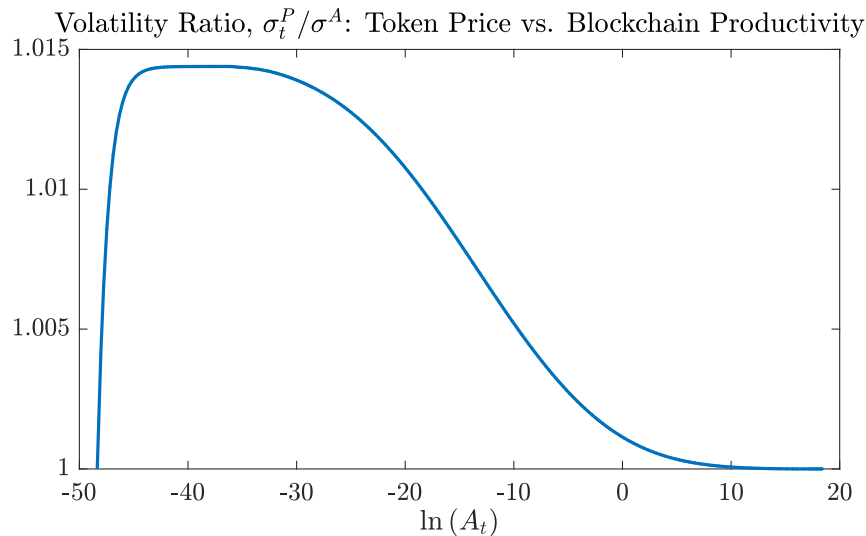


Figure 6: **Token Price Volatility Amplification.** This graph shows the ratio of token price volatility, σ_t^P , to blockchain productivity volatility, σ^A , which quantifies the strength of volatility amplification by the endogenous user adoption.

Price volatility dynamics. First notice that at full adoption $N_t = 1$, Equation (18) reveals that the ratio of P_t to A_t is a constant, and $\sigma_t^P = \sigma^A$. To the extent that the underlying productivity volatility σ^A is large (as suggested by our empirical analysis detailed in the appendix), token price volatility can be large even under full adoption.

In fact, σ_t^P is generally larger than σ^A as seen in Figure 6. The intuition is that a positive shock to A_t not only directly increases the demand for tokens but also increases adoption, which amplifies the shock’s impact on token price because N_t appears in the surplus flow.

Finally, Figure 6 also shows that token price volatility crucially depends on the adoption stage: it shoots up in the very early stage of adoption, gradually declines, and finally converges to σ^A as N_t approaches one.

7 Institutional Background

In this section, we clarify various concepts associated with cryptocurrencies. Importantly, we highlight two salient features shared among the majority of cryptocurrencies that our model captures: first, they are used as media of exchange on the platform (“token embedding”); second, their use exhibits network effects (“user-base externality”).

Blockchains, cryptocurrencies, and tokens. Blockchains are decentralized ledgers that record transactions and contracts that create a financial architecture for peer-to-peer interactions without relying on a centralized trusted third party. The blockchain technology potentially avoids a single point of failure and also reduces the concentration of market power, but still faces many challenging issues.¹⁶

Many blockchain applications feature cryptocurrencies and tokens. Bitcoin is the first widely-adopted decentralized cryptocurrency and over 1000 different “altcoins” (alternatives to Bitcoin) have been introduced over the past few years.¹⁷ Many altcoins such as Litecoin and Dogecoin are variants of Bitcoin, with modifications made to the original open-sourced protocol to enable new features. Others such as Ethereum and Ripple created their own blockchains and protocols to support their native currencies. In all these applications, cryptocurrencies serve as the media of exchange on their blockchain platforms.

Recently, initial coin offerings (ICOs) have also gained popularity. In ICOs, entrepreneurs sell “tokens” or “AppCoins” to investors. However, how tokens derive value is not clear, and thus, ICOs are facing quagmires regarding their legitimacies and distinctions from security issuances.¹⁸ In practice, tokens could represent (1) the claims on issuers’ cash-flows (“security tokens”), (2) the rights to redeem issuers’ products and services, or (3) the media of exchange among blockchain users.¹⁹ These tokens are often developed for blockchain applications (e.g., Gnosis and Golem) built on existing infrastructures (e.g., Ethereum or Waves). We focus on the majority of tokens issued thus far such as Filecoin, 0x, Civic, Raiden, and Basic Attention Token (BAT), which are the required media of exchange among platform users.

¹⁶Although Bank of England governor Mark Carney dismissed Bitcoin as an alternative currency, he recognized that the blockchain technology benefits data management by improving resilience by “eliminating central points of failure” and enhancing transparency and auditability while expanding what he called the use of “straight-through processes” including with smart contracts. In particular, “Crypto-assets help point the way to the future of money”. See, e.g., beat.10ztalk.com.

¹⁷Central banks, such as Bank of Canada, Monetary Authority of Singapore, and People’s Bank of China, are exploring the possibilities of digital currencies. In a controversial move, Venezuela announced in Feb 2018 the first digital currency, “petro”—an oil-backed token as legal tender for taxes and other public uses. Deutsche Bundesbank is working on the prototype of blockchain-based settlement systems for financial assets.

¹⁸See, for example, “Token Resistance,” *The Economist*, November 11th, 2017. In the recent hearing on Capital Markets, Securities, and Investment, March 14, 2018, the regulators also appear rather divided on the future of cryptocurrencies, digital currencies, ICOs, and Blockchain development.

¹⁹While the first ICO in 2013 raised a meager \$500k and sporadic activities over the next two years, 2016 saw 46 ICOs raising about \$ 100m and according to CoinSchedule, in 2017 there were 235 Initial Coin Offerings. The year-end totals came in over \$3 billion raised in ICO. In August, 2017, OmiseGO (OMG) and Qtum passed a US\$1 billion market cap today, according to coinmarketcap.com, to become the first ERC20 tokens built on the Ethereum network and sold via an ICO to reach the unicorn status.

Token embedding. Blockchain platforms often use native tokens to settle transactions among their users—a phenomenon that we refer to as “token embedding.” There are several reasons for this common practice.

First, transacting via a common token is convenient and saves costs of exchanging currencies. For example, the Ripple token (XRP) settles international payments on its blockchain at lower costs than the banking system. Most transactions and fundraising activities are carried out using Ethers (ETH) because of its convenience and popularity, despite the fact that the Ethereum platform allows other ERC-20 compatible cryptocurrencies

Second, for blockchain applications such as smart contracting, using a common unit of account eliminates the balance-sheet risk that would arise if assets and liabilities were denominated in different units of account (Doepke and Schneider (2017)).

Third, tokens provide incentives for entrepreneurs, programmers, and miners (or other ledger validators) who contribute to the platform.²⁰ For “proof-of-work” blockchains, such as Bitcoins, tokens are used to reward “miners” who maintain ledgers; owning tokens may allow users to maintain ledgers or provide services on other blockchains, such as Truebit (off-chain computation), OmiseGO (open payment), Livepeer (distributed video encoding), and Gems (decentralized mechanical Turk).

Fourth, introducing tokens allows the issuers to collect economic rent. In contrast to sovereigns who cannot easily commit to a money supply rule, blockchain developers can commit to an algorithmic rule of token supply to generate scarcity. Developers then collect rent through coin offerings—the fact that users can only conduct activities via the tokens of a blockchain platform generates the value of tokens, and thus, ICO revenues to the developers.

These rationales motivate us to focus on platforms with native tokens. But if the velocity of native tokens is infinite, their price is indeterminate. A key aspect of token embedding is that agents have to hold tokens for a certain period of time to conduct blockchain activities. First, reaching decentralized consensus takes time—the concept of finality time (Chiu and Koepl (2017)). Second, decentralized ledger maintainers often need to hold tokens as proof of the right to provide services.²¹ Third, smart contracts that automate contingent transfers

²⁰For example, as explained by Strategic Coin, BAT tokens serve as the medium of exchange among consumers, advertisers, and publishers who adopt the Brave web browser. The advertisers purchase ads using BAT tokens, which are then distributed to the publishers for hosting the ads and to the browser users (consumers) for viewing the ads.

²¹Proof-of-Stake protocols typically fall in this category.

often require escrowing the tokens. Finally, there are legal limits, e.g., the know-your-customer process in anti-money-laundering practice (e.g., Dfinity).

User-base externality. User-base externality has been well recognized as a defining features of P2P platforms and sharing economies. The utility of tokens increases, when more people use the blockchain platform, and thus, it is easier to meet transaction counterparties. Therefore, tokens represent assets that deliver convenience dividends (in terms of user transaction surplus) which increase in the size of platform user base.

This network effect is not restricted to blockchain platforms. It is particularly important on social and payment networks such as Facebook, Twitter, YouTube, WeChat, and PayPal. Other examples include club membership and collectibles of limited edition in sports. Therefore, the insights from our model broadly apply to platform currencies that can be (potentially) used in interactive online games (e.g., World of Warcraft), virtual worlds (e.g., Second Life), and sharing economies such as UBER and AirBnB.²²

8 Conclusion

We provide the first fundamentals-based dynamic pricing model of cryptocurrencies and platform tokens, taking into consideration the user-base externality and endogenous user adoption. Because the expectation of token price appreciation induces more agents to join the platform, tokens capitalize future user adoption, generally enhancing welfare and reducing user-base volatility.

Our model is flexible enough to admit multiple extensions. We outline four of them in Appendix C: 1. endogenous productivity growth; 2. cryptocurrency competition; 3. design of state-contingent token supply; 4. time-varying systematic risk of tokens. More generally, our neoclassical framework can be applied to the pricing of assets associated with a platform that features endogenous user base and network externality.

²²In fact, consistent with our model, when Tencent QQ introduced Q-coin, a case to which our model is applicable, many users and merchants quickly started accepting them even outside the QQ platform, tremendously accelerating adoption and token price appreciation. Annual trading volume reached billions of RMB in the late 2000s and the government had to intervene. See articles *China bars use of virtual money for trading in real goods* and *QQ: China's New Coin of the Realm?* (WSJ). Halaburda and Sarvary (2016) provide comprehensive discussions on various platform currencies.

Appendix A - Proofs

A1. Proof of Proposition 1

Figure 1 illustrates the determination of N_t given A_t and μ_t^P , which we take as a snapshot of the dynamic equilibrium with time-varying productivity and expectation of price change. The proof below takes the following steps. First, we show that given μ_t^P , there exists a \underline{A} such that for $A_t = A > \underline{A}$, the corresponding response curve,

$$R(n; A, \mu_t^P) = 1 - G(\underline{u}(n; A, \mu_t^P)) \quad (32)$$

$$= 1 - G\left(-\ln(n) + \ln\left(\frac{\phi}{A_t \alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right) \ln\left(\frac{1-\alpha}{r - \mu_t^P}\right)\right), \quad (33)$$

crosses the 45° line at least once in $(0, 1]$, and for any value of $A_t = A < \underline{A}$, the response curve never crosses the 45° line in $(0, 1]$. After proving the existence of $N_t > 0$ for $A_t \in [\underline{A}, +\infty)$, we prove the uniqueness given the increasing hazard rate of $g(u)$. Finally, we prove that N_t increases in μ_t^P . Before we start, for any $A_t = A > 0$, we define the value of its response function at $n = 0$: $R(0; A, \mu_t^P) = 0$. This is consistent with that given a zero user base, each agent derives zero transaction surplus from token holdings and chooses not to participate. Note that $\lim_{n \downarrow 0} R(n; A, \mu_t^P) = 0$, so the response function is continuous in n .

Given μ_t^P , we define a mapping, $A(n)$, from any equilibrium, non-zero value of user base, $n \in (0, 1]$, to the corresponding value of A_t , i.e., the unique solution to

$$1 - G\left(-\ln(n) + \ln\left(\frac{\phi}{A_t \alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right) \ln\left(\frac{1-\alpha}{r - \mu_t^P}\right)\right) = n, \quad \forall n \in (0, 1]. \quad (34)$$

This mapping is a continuous mapping on a bounded domain $\subseteq (0, 1]$. Then by the Least-Upper-Bound-Property of real numbers, the image set of this mapping, $\{A(n), n \in (0, 1)\}$, has an infimum, which we denote by \underline{A} . Now, for $A_t = \underline{A}$, consider a $n(\underline{A}) \in (0, 1]$ such that Equation (34) holds. For any $A > \underline{A}$, the LHS of Equation (34) is higher than the RHS, i.e., $R(n(\underline{A}); A, \mu_t^P) > n(\underline{A})$, so that the response curve of $A_t = A$ is above the 45° line at $n(\underline{A})$. Next, because the response function $R(n; A, \mu_t^P)$ is continuous in n and $R(1; A, \mu_t^P) \leq 1$ by definition in Equation (33), i.e., it eventually falls to or below the 45° line as n increases, there must exist a $n(A) \in (0, 1]$ such that when at $A_t = A$, Equation (34) holds by the

Intermediate Value Theorem. Therefore, we have proved that for any $A_t = A > \underline{A}$, there exists a non-zero user base. Throughout the proof, we fix μ_t^P , so \underline{A} is a function of μ_t^P .

Next, given $\frac{g(u)}{1-G(u)}$ is increasing, we show that the response curve crosses the 45° line exactly once when $A_t \in [\underline{A}, +\infty)$. First note that $R(n; A_t, \mu_t^P) - n$ either has positive derivative or negative derivative at $n = 0$. If it has positive derivative (i.e., the response curve shoots over the 45° line), then at n' , *the first time* the response curve crosses the 45° line again, the derivative of $R(n; A_t, \mu_t^P) - n$ must be weakly negative at n' , i.e., the response curve crosses the 45° from above,

$$g(\underline{u}(n'; A_t, \mu_t^P)) \frac{1}{n'} - 1 \leq 0. \quad (35)$$

Now suppose the response curve crosses the 45° line for the second time from below at $n'' > n'$, so the derivative of $R(n; A_t, \mu_t^P) - n$ at n'' must be weakly positive, and is equal to

$$\begin{aligned} g(\underline{u}(n''; A_t, \mu_t^P)) \frac{1}{n''} - 1 &= \frac{g(\underline{u}(n''; A_t, \mu_t^P))}{1 - G(\underline{u}(n''; A_t, \mu_t^P))} - 1 \\ &< \frac{g(\underline{u}(n'; A_t, \mu_t^P))}{1 - G(\underline{u}(n'; A_t, \mu_t^P))} - 1 \\ &= \frac{g(\underline{u}(n'; A_t, \mu_t^P))}{n'} - 1 \\ &< 0, \end{aligned} \quad (36)$$

where the first inequality comes from the increasing hazard rate and the fact that $\underline{u}(n; A_t, \mu_t^P)$ is decreasing in n for $n \in (0, 1]$, and the second inequality follows from (35) and the fact that the response curve crosses the 45° line at n' (i.e., $n' = R(n'; A_t, \mu_t^P) = 1 - G(\underline{u}(n'; A_t, \mu_t^P))$). This contradicts the presumption that the response curve reaches the 45° line from below (and the derivative of $R(n; A_t, \mu_t^P) - n$ is weakly positive). Therefore, we conclude that for $A_t \in [\underline{A}, +\infty)$, there exists a unique adoption level n . Now if $R(n; A_t, \mu_t^P) - n$ has negative derivative at $n = 0$, then in the previous argument, we can replace n' with 0 and show that there does not exist another intersection between the response curve and the 45° line beyond $n = 0$. Therefore, only if $R(n; A_t, \mu_t^P) - n$ has positive derivative at $n = 0$, do we have a positive (non-degenerate) adoption level.

Finally, we show that the non-degenerate adoption level, N_t , is increasing in μ_t^P . Consider

$\tilde{\mu}_t^P > \mu_t^P$. Suppose the contrary that their corresponding adoption levels satisfy $\tilde{N}_t \leq N_t$. Because we have proved that the response curve only crosses the 45° line only once and from above, given N_t , we have

$$1 - G\left(-\ln(n) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\mu_t^P}\right)\right) \geq n, \quad \forall n \in (0, N_t]. \quad (37)$$

We know that by definition,

$$\begin{aligned} \tilde{N}_t &= 1 - G\left(\underline{u}\left(\tilde{N}_t; A_t, \tilde{\mu}_t^P\right)\right) \\ &= 1 - G\left(-\ln\left(\tilde{N}_t\right) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\tilde{\mu}_t^P}\right)\right) \\ &> 1 - G\left(-\ln\left(\tilde{N}_t\right) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\mu_t^P}\right)\right) \\ &\geq \tilde{N}_t, \end{aligned} \quad (38)$$

where the first inequality uses $\tilde{\mu}_t^P > \mu_t^P$ and the second inequality uses the fact that $\tilde{N}_t \in (0, N_t]$ and the inequality (37). This contradiction implies that the adoption level N_t has to be increasing in μ_t^P .

A2. Derivation of the User-base Volatility

First, we consider the case without token. Using Itô's lemma, we can differentiate Equation (24) and then, by matching coefficients with Equation (28), derive μ_t^N and σ_t^N :

$$dN_t = -g\left(\underline{u}_t^{NT}\right) d\underline{u}_t^{NT} - \frac{1}{2}g'\left(\underline{u}_t^{NT}\right) \langle d\underline{u}_t^{NT}, d\underline{u}_t^{NT} \rangle, \quad (39)$$

where $\langle d\underline{u}_t^{NT}, d\underline{u}_t^{NT} \rangle$ is the quadratic variation of $d\underline{u}_t^{NT}$. Using Itô's lemma, we differentiate Equation (23)

$$\begin{aligned} d\underline{u}_t^{NT} &= -\frac{1}{N_t} dN_t + \frac{1}{2N_t^2} \langle dN_t, dN_t \rangle - \frac{1}{A_t} dA_t + \frac{1}{2A_t^2} \langle dA_t, dA_t \rangle \\ &= -\left(\frac{\mu_t^N}{N_t} - \frac{(\sigma_t^N)^2}{2N_t^2} + \mu^A - \frac{(\sigma^A)^2}{2}\right) dt - \left(\frac{\sigma_t^N}{N_t} + \sigma^A\right) dZ_t^A. \end{aligned} \quad (40)$$

Substituting this dynamics into Equation (39), we have

$$dN_t = \left[g(\underline{u}_t^{NT}) \left(\frac{\mu_t^N}{N_t} - \frac{(\sigma_t^N)^2}{2N_t^2} + \mu^A - \frac{(\sigma^A)^2}{2} \right) - \frac{1}{2} g'(\underline{u}_t^{NT}) \left(\frac{\sigma_t^N}{N_t} + \sigma^A \right)^2 \right] dt + g(\underline{u}_t^{NT}) \left(\frac{\sigma_t^N}{N_t} + \sigma^A \right) dZ_t^A, \quad (41)$$

By matching coefficients on dZ_t^A with Equation (28), we can solve for σ_t^N .

Next, we consider the tokenized economy. Once tokens are introduced, N_t depends on the expected token price appreciation μ_t^P , which is also a univariate function of state variable A_t because by Itô's lemma, μ_t^P is equal to $\left(\frac{dP_t/P_t}{dA_t/A_t} \right) \mu^A + \frac{1}{2} \frac{d^2 P_t/P_t}{dA_t^2/A_t^2} (\sigma^A)^2$. In equilibrium, its law of motion is given by a diffusion process

$$d\mu_t^P = \mu_t^{\mu^P} dt + \sigma_t^{\mu^P} dZ_t^A. \quad (42)$$

Now, the dynamics of \underline{u}_t becomes

$$d\underline{u}_t = -\frac{1}{N_t} dN_t + \frac{1}{2N_t^2} \langle dN_t, dN_t \rangle - \frac{1}{A_t} dA_t + \frac{1}{2A_t^2} \langle dA_t, dA_t \rangle - \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{1}{r - \mu_t^P} \right) d\mu_t^P - \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{1}{2(r - \mu_t^P)^2} \right) \langle d\mu_t^P, d\mu_t^P \rangle \quad (43)$$

Let σ_t^u denote the diffusion of \underline{u}_t . By collecting the coefficients on dZ_t^A in Equation (43), we have

$$\sigma_t^u = -\frac{\sigma_t^N}{N_t} - \sigma^A - \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{\sigma_t^{\mu^P}}{r - \mu_t^P} \right), \quad (44)$$

which, in comparison with Equation (40), contains an extra term that reflects the volatility of expected token price change. Note that, similar to Equation (39), we have

$$dN_t = -g(\underline{u}_t) d\underline{u}_t - \frac{1}{2} g'(\underline{u}_t) \langle d\underline{u}_t, d\underline{u}_t \rangle, \quad (45)$$

so the diffusion of N_t is $-g(\underline{u}_t) \sigma_t^u$. Matching it with the conjectured diffusion coefficient σ_t^N gives σ_t^N .

A3. Proof of $\underline{A}^{FB} < \underline{A}^{NT}$

To prove this inequality, consider the agent whose type is \underline{u}^{NT} , i.e., the type whose flow profit is equal to zero when $A_t = \underline{A}^{NT}$ in the tokenless economy. Therefore, we have the following

$$0 = N^{NT} \underline{A}^{NT} e^{\underline{u}^{NT}} \alpha \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha}{\alpha}} - \phi < \alpha \underline{A}^{NT} \bar{S} \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha}{\alpha}} - \phi, \quad (46)$$

where we use

$$\begin{aligned} N^{NT} e^{\underline{u}^{NT}} &= [1 - G(\underline{u}^{NT})] e^{\underline{u}^{NT}} < \int_{\underline{u}^{NT}}^{\bar{U}} e^{\underline{u}^{NT}} dG(u) + \int_{\underline{U}}^{\underline{u}^{NT}} e^u dG(u) \\ &< \int_{\underline{u}^{NT}}^{\bar{U}} e^u dG(u) + \int_{\underline{U}}^{\underline{u}^{NT}} e^u dG(u) \equiv \bar{S}. \end{aligned} \quad (47)$$

Recall that in the FB economy, we have

$$0 = \alpha \underline{A}^{FB} \bar{S} \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha}{\alpha}} - \phi. \quad (48)$$

By comparing the right expressions in the two preceding inequalities, we conclude $\underline{A}^{NT} > \underline{A}^{FB}$.

Appendix B - Parameter Choices

We choose the model parameters under the physical measure so that the model generates patterns that are broadly consistent with user adoption and token price dynamics.

We assume that capital markets are perfectly competitive. For simplicity, we price all assets including tokens via the following stochastic discount factor (“SDF”):

$$\frac{d\Lambda_t}{\Lambda_t} = -r dt - \eta d\widehat{Z}_t^\Lambda, \quad (49)$$

where r is the risk-free rate and η is the market price of risk for \widehat{Z}_t^Λ under the physical measure. Let ρ denote the correlation coefficient between the SDF and productivity A_t . With these assumptions, under the physical measure, A_t follows a GBM process, where the

drift coefficient, $\hat{\mu}^A$, is equal to $\mu^A + \eta\rho\sigma^A$ and the volatility coefficient is σ^A .

We use token price and blockchain user-base dynamics from July 2010 and April 2018. We normalize one unit of time in the model to be one year. Since we fix the token supply at M , the token price P_t completely drives the market capitalization (P_tM). We map P_t to the *aggregate* market capitalization of major cryptocurrencies.²³ Since we study a representative token economy, focusing on the aggregate market averages out idiosyncratic movements due to specificities of token protocols.

We collect the number of active user addresses for these cryptocurrencies and map the aggregate number to N_t . We normalize the maximum number of active addresses (in December 2017) to $N_t = 0.5$ and record its corresponding value of $\ln(A_t)$ in our model. We scale the number of addresses in other months by that of December 2017. With December 2017 as the reference point, we calculate the corresponding value of $\ln(A_t)$ for each month by applying the expected growth rate of A_t under the physical measure. As a result, we focus on the stage of adoption, i.e., $N_t \in [\underline{N}, 0.5]$, where $\underline{N} = 0.0001$.

Next, we choose parameter values such that the model generates data patterns in Figures 3 and 5. We set the annual risk-free rate, r , to 5% and choose $\mu^A = 2\% < r$ to satisfy the no-arbitrage restriction. As we have previously discussed, we interpret A_t as a process that broadly captures technological advances, regulatory changes, and the variety of activities feasible on the platform, all of which suggest a fast and volatile growth of A_t . This consideration motivates us to choose $\sigma^A = 200\%$. This parameter gives us both a high volatility for A_t but also much of the growth for A_t under the physical measure, as the physical-measure drift of A_t is $\hat{\mu}^A = \mu^A + \eta\rho\sigma^A$ (Girsanov's theorem).

To match the growth of N_t in the data, we set $\eta\rho = 1$, so that $\hat{\mu}^A = 202\%$ using the preceding equation. As a result, the user base N_t grows from $\underline{N} = 0.0001$ to 0.5 during the eight-year period of our data sample and the growth rate for the model-implied N_t matches that in data. One way to generate $\eta\rho = 1$ is to set η to 1.5, which is roughly the Sharpe ratio of ex-post efficient portfolio in the U.S. stock market (combining various factors) and ρ to 0.67, a sensible choice of betas for the technology sector (Pástor and Veronesi (2009)).

²³We include all sixteen cryptocurrencies with complete market cap and active address information on bitinfocharts.com: AUR (Auroracoin), BCH (Bitcoin Cash), BLK (BlackCoin), BTC (Bitcoin), BTG (Bitcoin Gold), DASH (Dashcoin), DOGE (DOGEcoin), ETC (Ethereum Classic), ETH (Ethereum), FTC (Feathercoin), LTC (Litecoin), NMC (Namecoin), NVC (Novacoin), PPC (Peercoin), RDD (Reddcoin), VTC (Vertcoin). They represent more than 2/3 of the entire crypto market.

We use the normalized distribution for u_i by truncating the Normal density function $g(u) = \sqrt{\frac{1}{2\pi\theta^2}} e^{-\frac{u^2}{2\theta^2}}$ within six-sigma on both sides. As the dispersion of u_i determines how responsive N_t is to the change of A_t , we match the curvature of N_t with respect to A_t by setting $\theta = 10/\sqrt{2}$, which implies that the cross-section variance of u_i is 50.

We set α to 0.3 so that the sensitivity of $\ln(P_t)$ with respect to N_t matches the data in the region where $N_t \in [\underline{N}, 0.5]$ as we show in Figure 5.

The remaining parameters quantitatively do not affect much the equilibrium dynamics. We set the participation cost, ϕ , to one and normalize M to 10 billion. As our model features monetary neutrality, P_t is halved when M is doubled but importantly the equilibrium dynamics is invariant.

Appendix C - Model Extensions

C1. Endogenous Growth: from User Base to Productivity

Our analysis thus far has taken the blockchain productivity process as exogenous. In reality, many token and cryptocurrency applications feature an endogenous dependence of platform productivity on the user base.

A defining feature of blockchain technology is the provision of consensus on decentralized ledgers. In a “proof-of-stake” system, the consensus is more robust when the user base is large and dispersed because no single party is likely to hold a majority stake; in a “proof-of-work” system, more miners potentially deliver faster and more reliable confirmation of transactions, and miners’ participation in turn depends on the size of user base through the associated media coverage (attention in general), transaction fees, and token price. More broadly, A_t represents the general usefulness of the platform. When more users participate, more types of activities can be done on the blockchain. Moreover, a greater user base potentially directs greater resources and research into the blockchain community, accelerating the technological progress.

The endogeneity of blockchain productivity and its dependence on the user base highlight the decentralized nature of this new technology. To reflect this fact and discuss its theoretical implications related to the growth and volatility amplification effects, we modify the process

of A_t as follows:

$$\frac{dA_t}{A_t} = (\mu_0^A + \mu_1^A N_t) dt + \sigma^A dZ_t. \quad (50)$$

By inspection of Equation (1), the definition of trade surplus, it appears that A_t and N_t are not separately identified from the perspective of individual users, because either of the two is simply part of the marginal productivity. However, this argument ignores the fact that by feeding N_t into the process of A_t , the growth rate of A_t is no longer i.i.d.

Consider the case where $d\mu^A(N_t)/dN_t > 0$. A higher level of N_t now induces faster growth of A_t , which leads to a higher level of N_t in the future. Similarly, a lower current level of N_t translates into a downward shift of the path of N_t going forward. In other words, the endogenous growth of A_t induces *persistence* in N_t . In our benchmark setting, N_t is reset every instant, depending on the exogenous level of A_t . Yet, here path dependence arises, which tends to amplify both the growth and unconditional volatility of N_t by accumulating and propagating shocks to A_t . A formal analysis of this extension is certainly important in light of improving quantitative performances of the model.

Another way to achieve such path dependence is to assume that agents' decision to join the community or quit incurs an adjustment cost, so N_t becomes the other aggregate state variable, just as in macroeconomic models where capital stock becomes an aggregate state variable when investment is subject to adjustment cost. However, such specification does not capture the endogenous growth of A_t .

C2. Alternative Tokens and Platform Competition

Many blockchain platforms accommodate not only their native tokens but also other cryptocurrencies. For example, any ERC-20 compatible cryptocurrencies are accepted on the Ethereum blockchain.²⁴ To address this issue, we may consider an alternative upper boundary of A_t . Define ψ as the cost of creating a new cryptocurrency that is a perfect substitute with the token we study because it functions on the same blockchain and therefore faces the same common blockchain productivity and agent-specific trade needs. This creates a reflecting boundary at \bar{A} characterized by a value-matching condition and a smooth-pasting

²⁴ERC-20 defines a common list of rules that all tokens or cryptocurrencies should follow on the Ethereum blockchain.

condition:

$$P(\bar{A}) = \psi \text{ and } P'(\bar{A}) = 0. \quad (51)$$

When token price increases to ψ , entrepreneurs outside of the model will develop a new cryptocurrency that is compatible with the rules of our blockchain system. So, the price level never increases beyond this value. Because it is a reflecting boundary, we need to rule out jumps of token prices, therefore the first derivative of $P(A_t)$ must be zero. Again we have exactly three boundary conditions for a second-order ODE and an endogenous upper boundary that uniquely pins down the solution.

Similarly, we may consider potential competing blockchain systems, and interpret ψ as the cost of creating a new blockchain system and its native token, which together constitute a perfect substitute for our current system. This creates the same reflecting boundary for token price. When token price increases to ψ , entrepreneurs outside of the model will build a new system.

Proposition 3 (Alternative Boundary). *The upper boundary condition is given by Equation (51) in the two following cases: (1) the blockchain system accepts alternative tokens or cryptocurrencies that can be developed at a unit cost of ψ ; (2) an alternative blockchain system that is a perfect substitute of the current system can be developed at a cost of ψ per unit of its native tokens.*

While our framework accommodates the effect of competition, a careful analysis of crypto industrial organization certainly requires more ingredients, especially those that can distinguish between the entry of multiple cryptocurrencies into one blockchain system and competing platforms/networks. Also constituting interesting future work is the impact of one platform using another platform's native tokens.

C3. Token Supply Schedule

In practice, many cryptocurrencies and tokens feature an increasing supply over time (for example, Bitcoin) or state-contingent supply in order to stabilize token price (for example, Basecoin). Our framework can be modified to accommodate this feature, and thus, serve as a platform for experimenting with the impact of token supply on user base growth and token price stability. For example, we may consider the law of motion of token supply M

given by an exogenous stochastic process such as

$$dM_t = \mu^M(M_t, N_t) dt + \sigma^M(M_t, N_t) dZ_t^A. \quad (52)$$

The Markov equilibrium then features two aggregate state variables, A_t and M_t .

An alternative formulation entails incrementals of M following Poisson-arrivals, as seen in Bitcoin's supply schedule. This formulation has the analytical advantage that equilibria between two Poisson arrivals still have only one state variable A_t . We can solve the model in a backward induction fashion, starting from the asymptotic future where token supply has plateaued and moving back sequentially in the Poisson time given the value function from the previous step.

As in many macroeconomic models, our framework features monetary neutrality: doubling the token supply *from now on* simply reduces token price by half and does not impact any real variables. However, neutrality is only achieved if the change of token supply is implemented uniformly and proportionally for any time going forward. If token supply is adjusted on a contingent basis, agents' expectation of token price appreciation is affected, through which supply adjustments influence user base, token demand, and the total trade surplus realized on the platform.

Finally, we emphasize that to achieve the desirable effects of a token supply schedule, the schedule must be implemented automatically without centralized third-party interventions, so that dispersed agents take the supply process as given when making decisions. Such commitment to rules and protocols highlights a key difference between cryptocurrency supply and money supply by governments – through the discipline of decentralized consensus, blockchain developers can commit to a token supply schedule.

C4. Bubbly Behavior of Token Price and Risk Premium

The expected token price appreciation under the physical measure is

$$\widehat{\mu}_t^P = \mu_t^P + \eta \rho \sigma_t^P. \quad (53)$$

The covariance between token price change and SDF shock, i.e., $\rho \sigma_t^P$, is priced at η . If the shock to blockchain productivity is orthogonal to SDF shock ($\rho = 0$), then $\widehat{\mu}_t^P = \mu_t^P$.

So far, we have fixed the correlation between SDF shock and shock to A_t as a constant. Yet as a blockchain platform or the general technology gains popularity, tokens become a systematic asset. Pástor and Veronesi (2009) emphasize that the beta of new technology tends to increase as it becomes mainstream and well adopted. We can allow the correlation between SDF and A_t to depend on N_t by decomposing the technological shock into two components under the physical measure,

$$d\widehat{Z}_t = \rho(N_t) d\widehat{Z}_t^\Lambda + \sqrt{1 - \rho(N_t)^2} d\widehat{Z}_t^I, \quad (54)$$

where the standard Brownian shock, $d\widehat{Z}_t^I$, is independent from the SDF shock, $d\widehat{Z}_t^\Lambda$. Therefore, the covariance between technological shock and SDF shock is $\rho(N_t)$, where we assume $d\rho/dN_t > 0$, that is the blockchain productivity shock becomes increasingly systematic as the user base grows. Under the risk-neutral measure, we have

$$\frac{dA_t}{A_t} = [\widehat{\mu}^A - \eta\rho(N_t)\sigma^A] dt + \sigma^A dZ_t. \quad (55)$$

Consequently, the risk-neutral, expected growth rate of A_t is $\widehat{\mu}^A - \eta\rho(N_t)\sigma^A$, which declines in N_t .

Therefore, as A_t grows, there are two opposing forces that drive P_t . On the one hand, the mechanisms that increase P_t are still there: when A_t directly increases the flow utility of token, or indirectly through N_t , token price increases. On the other hand, through the increase of N_t , the expected growth of A_t under the *risk-neutral* measure declines, which pushes P_t down. The former channel could dominate in the early stage of adoption while the channel of N_t -dependent token beta dominates in the later stage of adoption. A bubble-like behavior then ensues — P_t rises initially, and later as N_t rises, P_t declines because the risk-neutral expectation of A_t growth declines, resembling the formation and burst of a “bubble.”

References

- Athey, Susan, Ivo Parashkevov, Vishnu Sarukkai, and Jing Xia, 2016, Bitcoin pricing, adoption, and usage: Theory and evidence, *Working Paper*.
- Bakos, Yannis, and Hanna Halaburda, 2018, The role of cryptographic tokens and icos in fostering platform adoption, .
- Balvers, Ronald J, and Bill McDonald, 2017, Designing a global digital currency, *Working Paper*.
- Biais, Bruno, Christophe Bisière, Matthieu Bouvard, and Catherine Casamatta, 2017, The blockchain fold theorem, *Preliminary Work in Progress*.
- , and Albert Menkveld, 2018, Equilibrium Bitcoin Pricing, Working paper Toulouse School of Economics.
- Cao, Sean, Lin William Cong, and Baozhong Yang, 2018, Financial reporting and blockchains: Collaborative auditing, mis-statements, and regulation, *Working Paper*.
- Catalini, Christian, and Joshua S Gans, 2018, Initial coin offerings and the value of crypto tokens, Discussion paper, National Bureau of Economic Research.
- Chiu, Jonathan, and Thorsten V Koepl, 2017, The economics of cryptocurrencies—bitcoin and beyond, *Working Paper*.
- Chiu, Jonathan, and Tsz-Nga Wong, 2015, On the essentiality of e-money, Discussion paper, Bank of Canada Staff Working Paper.
- Cong, Lin William, and Zhiguo He, 2018, Blockchain disruption and smart contracts, *Forthcoming, Review of Financial Studies*.
- , and Jiasun Li, 2018, Decentralized mining in centralized pools, *Working Paper*.
- Doepke, Matthias, and Martin Schneider, 2017, Money as a unit of account, *Econometrica* 85, 1537–1574.
- Easley, David, Maureen O’Hara, and Soumya Basu, 2017, From mining to markets: The evolution of bitcoin transaction fees, *Working Paper*.
- Fernández-Villaverde, Jesús, and Daniel Sanches, 2016, Can currency competition work?, Discussion paper, National Bureau of Economic Research.
- Gandal, Neil, and Hanna Halaburda, 2014, Competition in the cryptocurrency market, *Working Paper*.
- Gans, Joshua S, and Hanna Halaburda, 2015, Some economics of private digital currency, in *Economic Analysis of the Digital Economy* . pp. 257–276 (University of Chicago Press).
- Halaburda, Hanna, and Miklos Sarvary, 2016, Beyond bitcoin, *The Economics of Digital Currencies*.
- Harvey, Campbell R, 2016, Cryptofinance, *Working Paper*.
- Huberman, Gur, Jacob Leshno, and Ciamac C. Moallemi, 2017, Monopoly without a monopolist: An economic analysis of the bitcoin payment system, working paper 17-92 Columbia Business School.

- Jackson, Lloyd K, 1968, Subfunctions and second-order ordinary differential inequalities, *Advances in Mathematics* 2, 307–363.
- Li, Jiasun, and William Mann, 2018, Initial coin offering and platform building, *Working Paper*.
- Nakamoto, Satoshi, 2008, Bitcoin: A peer-to-peer electronic cash system, *Online Publication*.
- Pagnotta, Emiliano, and Andrea Buraschi, 2018, An equilibrium valuation of bitcoin and decentralized network assets, Working paper Imperial College.
- Pástor, Luboš, and Pietro Veronesi, 2009, Technological revolutions and stock prices, *The American Economic Review* 99, 1451–1483.
- Raskin, Max, and David Yermack, 2016, Digital currencies, decentralized ledgers, and the future of central banking, Working Paper 22238 National Bureau of Economic Research.
- Saleh, Fahad, 2017, Blockchain without waste: Proof-of-stake, Discussion paper, working Paper.
- Sockin, Michael, and Wei Xiong, 2018, A model of cryptocurrencies, *Working Paper*.
- Yermack, David, 2017, Corporate governance and blockchains, *Review of Finance (Forthcoming)*.
- Yglesias, Matthew, 2012, Social cash: Could facebook credits ever compete with dollars and euros?, *Slate*.