

Sequential Learning, Asset Allocation, and Bitcoin Returns

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First draft: August 15, 2020

This version: November 10, 2021

Abstract

For optimal asset allocation, a mean-variance investor must learn about the joint dynamics of new and existing asset classes, not only their profitability. Bitcoin’s *digital gold* narrative provides a unique laboratory to test this hypothesis. We find that a decrease in investors’ estimate on correlation between Bitcoin and the US stock markets strongly predicts higher Bitcoin returns the next day. The same empirical pattern universally appears in out-of-sample predictions, global equity markets, and other cryptocurrencies. Our stylized model and empirical proxy for Bitcoin demand explain the predictability pattern in light of asset allocation practices and investors’ learning on time-varying correlation.

Key words: Bitcoin, Uncertainty, Learning, Time-Varying Correlation, Return Predictability

JEL Classification: G12, G15, D83

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1 Introduction

Investment is a never-ending battle against risk and uncertainty. Investors routinely quantify risk in existing assets while dealing with uncertainty in new assets. Then what would investors do if an emerging asset never belongs to any existing asset class? What if the asset’s narrative suddenly changes (Shiller 2020) and uncertainty escalates because of the change? Will investors’ response to a new narrative leave any trail on asset prices?

Bitcoin’s narrative evolution provides a perfect laboratory to answer those questions. Bitcoin begins as a decentralized peer-to-peer *payment system* (Nakamoto 2008), and it explodes as a speculative digital *currency* until late 2017. Since then, many Bitcoin advocates promote a new narrative, *digital gold*, that diverts attention from slow and expensive Bitcoin transactions (Hinzen et al. 2020).¹ The new narrative finally persuades traditional investors, such as pension funds and sovereign wealth funds, to include Bitcoin in their portfolios for diversification benefits.² However, investors face unprecedented uncertainty in Bitcoin’s correlations with other assets in their portfolios. For example, the conditional correlation between Bitcoin and the US stock markets begin to fluctuate since the inception of Bitcoin futures markets in late 2017.³

Such a structural break in correlation dynamics allows us to test our hypothesis: rational investors keep rebalancing their Bitcoin-stock portfolios by tracking the time-varying correlation of Bitcoin and their existing well-diversified stock portfolios. If so, investors’ learning, or equivalently changes in their beliefs, may systematically affect Bitcoin demands and prices.⁴ Using predictive regressions, we find that an increase (decrease) in investors’ estimate on correlation between Bitcoin and S&P500 predicts lower (higher) Bitcoin returns the next day.⁵ However, the level of correlation fails to predict subsequent Bitcoin returns.

This predictability evidence is puzzling in two aspects. First, in many cases, predictability appears to exist only by levels rather than changes because of a bias in regressions with persistent predictor variables (Stambaugh 1999). Second, the sign of Bitcoin return predictability contradicts

¹See Hardle et al. (2020) for an extensive literature review.

²See news articles at bit.ly/3qxaVmt, bloom.bg/3gl5zG2, bloom.bg/3qCcoIc, on.wsj.com/3htc369, and <https://on.wsj.com/37clZgh>.

³This is ironic because Bitcoin’s diversification benefit becomes suddenly uncertain as soon as traditional investors participate in the markets while recognizing Bitcoin as a mainstream asset.

⁴See Bianchi et al. (2020) for other time-varying aspects of Bitcoin.

⁵We have 765 observations, R^2 is above 1%, and Bitcoin’s daily return volatility is 5%, all of which are comparable to the market return predictability results in the literature with 60 years of monthly data.

conventional wisdom. A high correlation means a small diversification benefit; therefore, the risk premium should be high enough to compensate it. However, we observe lower subsequent returns after an increase in correlation.

Using a simple intuition, we argue that this puzzling predictability is a footprint of investors' learning behavior and trading practices. Suppose a fund manager runs a portfolio of Bitcoin and an S&P500 index fund. She revises her estimates on inputs for portfolio optimization at market closing. She requests in-house traders or external trading firms to rebalance her portfolio accordingly. Then the traders would split and delay orders during the next trading day hoping for executing at better prices or hiding the fund manager's intentions (Kyle 1985, Admati and Pfleiderer 1989).⁶ Therefore, *asynchronous portfolio rebalancing*—time difference between rebalancing decisions and order executions—can generate return predictability at daily frequency: an increase in correlation estimate today leads to a downward price pressure tomorrow.⁷ This lead-lag relationship is observable to econometricians but negligible to traders who face huge intra-day volatility.

To verify our empirical findings and intuition, we construct empirical proxies for investors' daily Bitcoin demand due to their learning on correlation in the joint dynamics of Bitcoin and S&P500. We fit a model of Dynamic Conditional Correlation (DCC) Generalized Autoregressive Conditional Heteroskedasticity (GARCH) introduced by Engle (2002). Then we compute Bitcoin demand proxies as conditional optimal portfolio weights on Bitcoin from the standard mean-variance portfolio theory (Markowitz 1952). Our proxies for Bitcoin demand uncover important empirical facts that conditional correlation estimates alone cannot reveal.

Using the proxies for Bitcoin demand, we find that a one-standard-deviation increase (decrease) in the daily Bitcoin-demand-change due to learning on correlation can predict 0.5% higher (lower) subsequent Bitcoin returns with Newey-West adjusted t-statistics above 4.0. Other existing predictors or extensive control variables barely affect the magnitude or the statistical significance of the coefficient estimate. In addition, several well-known Machine Learning algorithms show that our predictor is the most dominant among the predictors documented in the literature even if non-

⁶See an article related to this Bitcoin trading behavior at <https://yhoo.it/3i9ctPO>. Also, see empirical evidence on stealth trading (Barclay and Warner 1993, Chakravarty 2001), order imbalance (Chan and Fong 2000, Barber et al. 2008), and institutional investors (Sias and Starks 1997, Keim and Madhavan 1995).

⁷Our explanation does not require that all investors rebalance their portfolios every day. See Section 6.5 for a related discussion.

linearity and interactions are considered.⁸ Finally, out-of-sample R^2 by robust linear models can rise to 2.3% at daily frequency (3.5% before COVID-19).

Our intuition on Bitcoin’s return-predictability produces five groups of testable predictions, and we find supportive evidence in all of them. First, the same empirical patterns of Bitcoin’s return-predictability universally appear in seven major international equity markets plus Ethereum (ETH) and Ripple (XRP). Also, aggregating global Bitcoin demands across countries improves predictability. Second, signals from time-varying volatilities are too noisy for investors to use for portfolio optimization. This finding explains why changes in volatility estimates fail to predict Bitcoin returns. Third, changes in Bitcoin demand proxies are superior to changes in correlation estimates, in terms of prediction performance, supporting our Bitcoin demand channel. Fourth, Bitcoin’s return-predictability is conditional on the levels of volatilities and risk premia, consistent with the implications of portfolio optimization. Lastly, our predictor variable shows no predictability prior to the introduction of Bitcoin futures, when correlation is practically constant, consistent with our intuitive explanation based on investors’ learning.

Furthermore, by extending our empirical proxy construction, we build a stylized static model of Bitcoin prices to quantitatively investigate the Bitcoin return predictability. With reasonably calibrated parameters, the model can simultaneously match the observed coefficient estimates on the changes and levels of correlations in predictive regressions. Therefore, all evidence supports our intuition that investors’ learning on correlation can explain the puzzling patterns of Bitcoin return predictability in the presence of asynchronous portfolio rebalancing.

The contribution of the paper is threefold. First, this paper proves that Bitcoin and potentially other cryptocurrencies are closely linked to mainstream assets, as opposed to the literature that implicitly isolates such a new asset class from existing ones. We highlight (1) the ex-ante diversification benefits of Bitcoin for stock investors and (2) the importance of uncertainty and investors’ learning in the joint dynamics with stock markets, not only in the Bitcoin’s own dynamics.⁹ Second, this paper discovers a rational aspect of Bitcoin prices through the impact of

⁸LASSO, Group Lasso, Elastic Net, Random Forest, Gradient Boosting Machine, and Neural Network with one hidden layer. The last four algorithms consider non-linearity and all possible interactions among predictors.

⁹Several researches estimate Bitcoin’s time-varying correlations and suggest an effective portfolio strategy to investors in practice. See, for example, [Klein et al. \(2018\)](#) among many others. Also, [Gao and Nardari \(2018\)](#) perform a similar task on commodities. However, none of them study how investors actually react to time-varying correlations and how the investors’ reactions affect Bitcoin prices.

passive institutional investors, in contrast to the existing literature that focuses mainly on Bitcoin’s irrational side driven by retail investors.¹⁰ We find, both empirically and theoretically, that the puzzling sign of Bitcoin return-predictability can arise from investors’ rational behavior: portfolio optimization with learning. Lastly but most importantly, we answer a general economic question: how does investors’ learning affect asset prices?

The paper proceeds as follows. Section 2 presents motivating and puzzling empirical findings on Bitcoin return predictability. Section 3 provides an intuitive explanation for the puzzle and constructs an empirical proxies for Bitcoin demand. Section 4 validates Bitcoin return predictability from various angles. Section 5 investigates testable predictions from our intuitive explanation. Section 6 develops a stylized static model of Bitcoin prices and verifies our intuition on the Bitcoin’s return-predictability puzzle quantitatively. Section 7 concludes.

2 Motivation: Time-Varying Correlations and Bitcoin Returns

This section presents a motivating empirical finding. First, we explain how to estimate the time-varying correlation between Bitcoin and stock market returns. Next, we show a structural break of the correlation dynamics. Finally, we establish a novel but puzzling evidence of Bitcoin return predictability.

2.1 Estimation of Time-Varying Correlations

We choose a parsimonious time-series model to describe the joint dynamics of Bitcoin and the stock markets. Our workhorse model for time-varying correlations is DCC(1,1)-GARCH(1,1), Dynamic Conditional Correlation – Generalized Autoregressive Conditional Heteroskedasticity, by Engle (2002). Nevertheless, we find that our results are robust to different model choices. See Section A.1 in the online appendix for alternative multivariate GARCH models with conditional correlations.

Let a 2×1 vector $\mathbf{r}_t = [r_{b,t} \ r_{m,t}]^\top$ denote the daily log returns of Bitcoin and S&P500 index at time t .

$$\mathbf{r}_t = \boldsymbol{\mu} + \mathbf{e}_t \quad \text{where} \quad \mathbf{e}_t \sim N(\mathbf{0}, \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t),$$

¹⁰For example, consider price manipulations or retail investors’ behavioral mistakes. See Cheng et al. (2019), Makarov and Schoar (2020), and Griffin and Shams (2020) among many others.

where $\mathbf{D}_t = \text{diag}\{\sigma_{b,t}, \sigma_{m,t}\}$ is a diagonal matrix of the time-varying conditional volatilities, modeled by univariate GARCH(1,1):

$$\sigma_{i,t}^2 = \omega_i + \alpha_i e_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2 \quad \text{for } i = b, m,$$

while \mathbf{R}_t is the conditional correlation matrix of \mathbf{e}_t , modeled by DCC(1,1):

$$\mathbf{R}_t = \mathbf{Q}_t^{*-1} \mathbf{Q}_t \mathbf{Q}_t^{*-1} \quad \text{where } \mathbf{Q}_t = (1 - a - b) \overline{\mathbf{Q}} + a \boldsymbol{\epsilon}_{t-1} \boldsymbol{\epsilon}_{t-1}^\top + b \mathbf{Q}_{t-1}, \quad (1)$$

where $\overline{\mathbf{Q}}$ is the unconditional covariance matrix of $\boldsymbol{\epsilon}_t = \mathbf{D}_t^{-1} \mathbf{e}_t \sim N(\mathbf{0}, \mathbf{R}_t)$ and \mathbf{Q}_t^* is a diagonal matrix whose diagonal elements are the square roots of the diagonal elements of \mathbf{Q}_t .¹¹ The model reduces to the Constant Conditional Correlation (CCC) GARCH model of [Bollerslev \(1990\)](#) if \mathbf{R}_t is time-invariant $\mathbf{R}_t = \overline{\mathbf{R}}$ and so $\mathbf{Q}_t = \overline{\mathbf{Q}}$, which will produce $a \approx 0$ in estimation.

2.2 A Structural Break in Correlations Dynamics

In late 2017, Bitcoin experiences an iconic event that attracts attention from mainstream investors; Bitcoin futures markets begin at Chicago Board Options Exchange (CBOE) on December 11 and at Chicago Mercantile Exchange (CME) on December 18, 2017.¹² Since then the nature of uncertainty in Bitcoin changes from idiosyncratic to systematic ([Pástor and Veronesi 2009](#)), and Bitcoin trading and price dynamics change significantly, as documented in the literature (for example, [Augustin et al. 2020](#), [Hardle et al. 2020](#), and [Kim et al. 2020](#)).

To see how the correlation dynamics change around this iconic event, we split the full sample (from January 2015 to December 2020) into two so that the second period sample starts on Dec 18, 2017. The resulting sample sizes for two sub-samples are 746 and 765, respectively.¹³

The estimated conditional correlations in our dataset display a dramatic structural break in the correlation dynamics. Table 1 shows the parameter estimates of different multivariate GARCH models for three periods: 1) from January 2015 to December 2020, 2) from January 2015 but prior to the Bitcoin futures markets (pre-futures sample, henceforth), and 3) from the beginning

¹¹To ensure model validity, parameters a and b satisfy the restrictions $a \geq 0$, $b \geq 0$, $a + b < 1$, and \mathbf{Q}_t should be positive definite.

¹²CBOE delisted Bitcoin futures in 2019 temporarily but CME never did.

¹³Our results are robust to different choices on split date around this date.

of Bitcoin futures markets to December 2020 (post-futures sample, henceforth). In the table, the parameter a in DCC(1,1)-GARCH(1,1) model is zero in the pre-futures sample but 0.030 (t-stat 2.01) in the post-futures sample. The same pattern appears in the other multivariate GARCH specifications. The estimated conditional correlations (dashed line) in Figure 1 are practically zero and flat for the pre-futures sample while the correlations fluctuate for the post-futures sample. That is, the time-variation of correlation comes entirely from the post-futures sample.

If investors constantly monitor the correlation dynamics, they should notice the structural break at certain point. To see when the investors first notice the structural change, we perform rolling-window estimations starting May 2015. Figure A.2 in the online appendix shows that investors' real-time perception on correlation dynamics changes from "constant" to "time-varying" right after Bitcoin futures markets begin. This real-time estimation result is consistent with our learning-based explanation in Section 3.1.

Co-occurrence of the structural break and futures trading is interesting and deserves deeper investigations in both theoretical and empirical aspects. However, the results and conclusion of our analysis do not depend on the causal link between the structural break and futures trading. Therefore, finding the root cause of the structural break is beyond the scope of this paper, let alone various forces that exist behind the observed structural break.¹⁴ Instead, we focus on the Bitcoin return predictability for the period when correlation is time-varying.

2.3 Bitcoin's Return-Predictability Puzzle

The introduction of futures markets is Bitcoin's big leap toward a mainstream asset. Institutional investors who are fueled by the *digital gold* narrative of Bitcoin start to include Bitcoin in their portfolios. Such passive investors' main goal is diversification rather than speculation, yet they still need to estimate correlations for portfolio optimization. Therefore, the correlation affects passive investors' demand and equilibrium Bitcoin prices. We argue that such effects can be empirically detectable because institutional investors become major players in the Bitcoin markets lately. The time-varying correlations in the post-futures period offers an opportunity to test this idea.

¹⁴For example, changes in overall money supply or fund flows can generate positive correlations across different asset classes. Co-varying risk appetites or speculative sentiments can do the same. Yet, a rising demand of safe haven assets can generate negative correlations. If each of these factors acts dominantly in different times, the correlation between Bitcoin and the stock market can be time-varying. Furthermore, investors' learning on the correlation itself might create the time-variation of correlations as well.

In fact, correlations between asset returns are key elements in canonical asset pricing models. The time-varying correlation can imply a time-varying Capital Asset Pricing Model (CAPM) beta in a Conditional CAPM or a state variable in an Intertemporal CAPM. Therefore, the traditional asset pricing models suggest that time-varying CAPM betas or correlation levels should explain the time-variation of Bitcoin risk premium and predict subsequent Bitcoin returns.

However, we find that none of these variables explain Bitcoin’s risk premium. Table 2 shows that these variables fail to predict subsequent Bitcoin returns when we construct the variables from the DCC-GARCH estimation at daily frequency.¹⁵ Instead, changes in correlation strongly predict Bitcoin returns. The estimated slope coefficient is -0.153, and its Newey-West adjusted t-statistic is -3.7 with an R^2 of 1.05%. This predictability is robust to various specifications: excluding samples after the COVID-19 outbreak, using robust linear models such as LAD (Least Absolute Deviation) and Rank regressions, transforming the data by Inverse Normal Transformation, or trimming the data. In all alternative specifications, the predictability evidence is stable or even stronger than the baseline case. Also, our *post-futures* sample size is large enough to provide reliable evidence. We have 765 observations, R^2 is above 1%, and Bitcoin’s daily return volatility is 5%, all of which are comparable to the market return predictability results in the literature with 60 years of monthly data.¹⁶

This predictability evidence is puzzling. First, the level of correlation cannot predict subsequent Bitcoin returns whereas changes in correlation can. In many empirical cases, the opposite is often observed because of the bias in regressions with persistent predictor variables. Second, even if we attribute the lack of predictability to data or econometric issues, the sign of predictability by changes in correlation contradicts conventional wisdom and what canonical models suggest. High correlation means a small diversification benefit and so the risk premium should be high enough to compensate it. However, we observe lower returns right after an increase in correlation.¹⁷ We provide an intuitive explanation for this puzzle in the next section while formalizing it in an equilibrium model in Section 6.

¹⁵Time-varying CAPM beta estimates of Bitcoin are constructed from DCC-GARCH estimates on conditional correlation and volatilities.

¹⁶Bitcoin’s high volatility can be also interpreted as a subordinated process: Bitcoin behaves as if the clock for Bitcoin runs fast. This interpretation also supports the use of daily returns data for a short period.

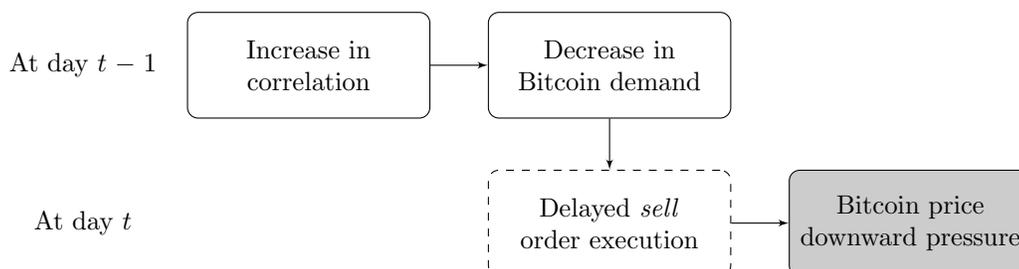
¹⁷A level of correlation tends to be high when changes in correlation is high because $cov(\rho_t, \Delta\rho_{(t-1):t}) = cov(\rho_t, \rho_t - \rho_{t-1}) = var(\rho_t) - cov(\rho_t, \rho_{t-1}) = var(\rho_t)(1 - cor(\rho_t, \rho_{t-1})) > 0$

3 Asset Allocation with A Mean-Variance Perspective

This section provides a simple intuition on the predictability and its sign. Then we show how to construct an empirical proxy for such a Bitcoin demand due to time-varying correlations. We also confirm empirically that Bitcoin provides diversification benefits to passive institutional investors.

3.1 Intuition: Asynchronous Portfolio Rebalancing as Trading Practice

Suppose a *passive* fund manager revises her estimates on inputs for portfolio optimization at market close. Then she requests in-house traders or external trading firms to rebalance her portfolio. The traders in practice rarely execute orders immediately. Instead, they split and delay orders during the next trading session while hoping for better execution prices without revealing the fund manager's intentions (Kyle 1985, Admati and Pfleiderer 1989).¹⁸ At an aggregate level, such delayed rebalancing demands are highly correlated across passive investors who observe the same public information such as historical prices. Therefore, an increase in correlations today suppresses the demand for Bitcoin the next day and Bitcoin prices. Likewise, a decrease in correlation today can explain an upward pressure in Bitcoin prices the next day. This simple intuition explains the puzzling sign of Bitcoin return predictability which is observable to econometricians but ignorable to traders who face huge intra-day volatility. This one-day time lag in rebalancing is so negligible at monthly frequency that conventional empirical tests of canonical asset pricing models often overlook this short-term predictability.¹⁹ The following flowchart visualizes the mechanism behind the Bitcoin predictability.



This intuition on the daily return predictability is also supported by rich empirical evidence.

¹⁸See an article related to this Bitcoin trading behavior at <https://yhoo.it/3i9ctPO>.

¹⁹See discussion in Section 6.5 for the validity of assumption on daily rebalancing.

Stealth trading is prevalent in the market (Barclay and Warner 1993, Chakravarty 2001), order imbalance predicts daily returns (Chan and Fong 2000, Barber et al. 2008), and institutional investors' behaviors create return autocorrelation at daily level (Sias and Starks 1997, Keim and Madhavan 1995). Furthermore, this learning-(asynchronous)-rebalancing mechanism is valid even if investors rebalance less frequently; for example, they maybe rebalance once a month but on a different day in a month so that there are always some investors who rebalance their portfolios on any given day. We provide a further discussion in Section 6.5.

We argue that our intuition is more than a plausible storytelling. First, we formalize our intuitive explanation and develop a static model of Bitcoin prices in Section 6. We show that the model can replicate the puzzling return predictability evidence with reasonable parameter values. Second, the proposed mechanism leads us to multiple predictions that are empirically testable; Section 5 investigates five different groups of predictions in detail. For example, the intuition implies that the return predictability must be observable only when the correlation is significantly time-varying and so investors have strong incentives to learn. In other words, we should observe absence of predictability when the correlation is practically constant, as shown in Section 5.5 as a placebo test. Finally, we discuss why the return predictability results are difficult to explain in other ways.

Alternative explanation #1: time-varying risk premium The Bitcoin return predictability puzzle is difficult to explain by the time-varying risk premium in a traditional asset pricing framework with investors' learning. First, it is empirically and conceptually problematic to construct an economically meaningful proxy for risk premium at daily frequency. Yet, using monthly returns leaves us with only 36 observations because we have only three years of data since the correlations start to fluctuate. Second, more importantly, Bitcoin does not belong to a traditional economic framework. Diversity of Bitcoin pricing models in the literature is a proof. Numerous non-standard components are used in the recent Bitcoin pricing literature, e.g., gradual adoption, Bitcoin mining cost, payment transaction volume, size of network, halving, and sentiments.²⁰ Ignoring these exotic forces is unrealistic, yet unifying all of them is not only challenging but creating more confusion.

²⁰See Athey et al. 2016), Biais et al. (2020), and Cong et al. (Forthcoming) among many others.

Alternative explanation #2: time-varying sentiment One might suggest that changes in correlation is a proxy for other variables such as investors' sentiment level. This type of conjecture is hard to prove or disprove, but we find this is unlikely the case. Variables like common sentiments, excessive liquidity, or degree of disagreement are highly persistent at daily frequency, yet changes in correlation are barely serially-correlated, by construction. Section 4.2.2 and 4.2.3 show that the predictability results are unaffected by various sentiment proxies in predictive regressions and machine learning approaches.

3.2 Empirical Demand Proxy: Construction and Decomposition

Our simple intuition is based on the assumption that passive institutional investors recognize Bitcoin as an investment asset that provides additional diversification benefits. Since those investors are often short-sale constrained, they must exclude Bitcoin if the optimal weight on Bitcoin in their portfolios is negative. Therefore, our intuition will lose ground if the weights in our sample period are mostly negative. To rule out such a possibility, we first compute the time-varying optimal weights on Bitcoin in a passive portfolio as an empirical proxy for Bitcoin demand. These optimal weights turn out mostly positive implying that passive institutional investors' demands for Bitcoin exist and vary over time. Also, using these weights as an empirical proxy helps us understand and explain the return predictability by a simple mean-variance framework.

3.2.1 Construction of a Proxy

Suppose there are mean-variance investors who want to maximize the Sharpe ratio of their investment portfolios.²¹ They mix a new asset, Bitcoin, with their existing well-diversified risky portfolio such as stock market portfolios, say, S&P500 index funds. We let $\mu_{b,t}$ and $\mu_{m,t}$ denote the conditional (subjective) risk premia of Bitcoin and the stock market, respectively. Similarly, $\sigma_{b,t}$ and $\sigma_{m,t}$ denote the conditional (subjective) volatilities of Bitcoin and the stock market, respectively. Then the investors' ex-ante optimal weight on Bitcoin in the new tangency portfolio is given by

$$w_{b,t} = \frac{\sigma_{m,t}^2 \mu_{b,t} - \rho_t \sigma_{b,t} \sigma_{m,t} \mu_{m,t}}{\sigma_{m,t}^2 \mu_{b,t} - \rho_t \sigma_{b,t} \sigma_{m,t} \mu_{m,t} + \sigma_{b,t}^2 \mu_{m,t} - \rho_t \sigma_{b,t} \sigma_{m,t} \mu_{b,t}} = \frac{\mu_t^* - \rho_t \sigma_t^*}{(\mu_t^* - \rho_t \sigma_t^*) + (\sigma_t^* - \rho_t \mu_t^*) \sigma_t^*}, \quad (2)$$

²¹That is, a riskfree asset exists in the market without a borrowing constraint so that the investors can maximize their utility function with their own risk aversion by mixing the tangency portfolio and the riskfree asset.

where ρ_t is the correlation coefficient between Bitcoin and the stock market, and where $\mu_t^* = \mu_{b,t}/\mu_{m,t}$ and $\sigma_t^* = \sigma_{b,t}/\sigma_{m,t}$ are Bitcoin’s risk premium ratio and volatility ratio relative to the stock market, respectively.²²

We regard this optimal weight $w_{b,t}$ as an empirical proxy for an investors’ Bitcoin demand, which is a function of $(\mu_t^*, \sigma_t^*, \rho_t)$, that is, $w_{b,t} = w_b(\mu_t^*, \sigma_t^*, \rho_t)$. Intuitively, $w_{b,t}$ monotonically decreases with ρ_t and σ_t^* within empirically reasonable ranges.²³ Note that all the parameter values in the equation above are the investor’s point estimates under her subjective belief, rather than true values. Also, we assume that the investors have anticipated utility (Kreps 1998, Cogley and Sargent 2008) with which an agent sequentially estimates and updates the parameters but treat them as true values at the moment without uncertainty for decision making. Therefore, true parameter values of $(\mu_t^*, \sigma_t^*, \rho_t)$ are irrelevant to proxy construction as long as we can approximate investors’ belief on them.

Finally, we stress that we are building empirical proxies rather than an equilibrium asset pricing model here although our approach relies on several assumptions that are often used in equilibrium models. After all, constructing a demand proxy provides us with a quantitative framework to investigate the Bitcoin return predictability puzzle in the remaining sections.

3.2.2 Decomposition of a Proxy

Our proxy for Bitcoin demand is based on investors’ subjective learning about the inputs $(\mu_t^*, \rho_t, \sigma_t^*)$ for portfolio optimization. In Bitcoin markets, not all investors fully perform Bayesian learning. Furthermore, what investors would learn depends on their investment motives because their attention and resources are limited in the real world. If an investor trades Bitcoin for price speculation (speculative investors, hereafter), she would barely care about diversification benefits. The investor rather directs all her attention and resources to learn about μ_t^* , not (ρ_t, σ_t^*) . If an investor manages a passive portfolio (non-speculative investors, hereafter), her focus is to learn about (ρ_t, σ_t^*) rather than μ_t^* because she believes that predicting highly volatile Bitcoin’s price is a difficult task and only adds extra errors to the portfolio optimization.

This simple view on two investors with distinct investment motives helps us refine our analysis

²²Refer to any college level textbook on investments for (2).

²³We provide formal conditions for demand monotonicity in online appendix A.2.

by decomposing our proxy for Bitcoin demand. First, inspired by speculative investors' motive, we define $w_{b,t}^{(mean)} \triangleq w_{b,t}(\mu_t^*, \bar{\sigma}^*, \bar{\rho})$ as the optimal weight on Bitcoin when investors learn only on μ_t^* while substituting the unconditional estimates $(\bar{\sigma}^*, \bar{\rho})$ for (σ_t^*, ρ_t) . On the other hand, $w_{b,t}^{(c+v)} \triangleq w_b(\bar{\mu}^*, \sigma_t^*, \rho_t)$ is the optimal weight on Bitcoin when passive (non-speculative) investors learn only on (σ_t^*, ρ_t) while substituting the unconditional estimate $\bar{\mu}^*$ for μ_t^* . Next, we decompose the conditional Bitcoin demand $w_{b,t}$ as follows, based on a discrete-time-version analogy to a total derivative of $w_{b,t}$ with respect to (ρ_t, σ_t^*) and μ_t^* .

$$w_{b,t} - \bar{w}_b \approx \underbrace{[w_{b,t}^{(c+v)} - \bar{w}_b]}_{\text{Non-speculative}} + \underbrace{[w_{b,t}^{(mean)} - \bar{w}_b]}_{\text{Speculative}}, \quad (3)$$

where $\bar{w}_b = w_b(\bar{\mu}^*, \bar{\sigma}^*, \bar{\rho})$ is the unconditional (average) Bitcoin demand. The first term is related to diversification benefits whereas the second term is mainly about the subjective outlook on Bitcoin prices. Thus we interpret the first term as non-speculative demand for Bitcoin by investors' learning while the second term as speculative demand. Next, we can further decompose the first term in Equation (3), according to its source of learning:

$$w_{b,t}^{(c+v)} - \bar{w}_b \approx \underbrace{[w_{b,t}^{(cor)} - \bar{w}_b]}_{\text{from correlation}} + \underbrace{[w_{b,t}^{(vol)} - \bar{w}_b]}_{\text{from volatility}}, \quad (4)$$

where $w_{b,t}^{(cor)} = w_{b,t}(\bar{\mu}^*, \bar{\sigma}^*, \rho_t)$ and $w_{b,t}^{(vol)} = w_{b,t}(\bar{\mu}^*, \sigma_t^*, \bar{\rho})$ are the optimal weights on Bitcoin when investors learn either correlation only or volatility ratio only, respectively. For example,

$$\Delta w_{b,(t-1):t}^{(cor)} = w_{b,t}^{(cor)} - w_{b,t-1}^{(cor)}, \quad (5)$$

where $w_{b,t}^{(cor)} = w_{b,t}(\bar{\mu}^*, \bar{\sigma}^*, \rho_t)$. That is, the first term in (4) is the time-varying Bitcoin demand due to correlation changes while the second term is due to volatility changes. Later, this further decomposition allows us to empirically identify what passive non-speculative investors mainly learn. Finally, we decompose the Bitcoin demand change $\Delta w_{b,(t-1):t}$ as follows by combining Equation (3) and (4).

$$\Delta w_{b,(t-1):t} \triangleq w_{b,t} - w_{b,t-1} \approx \Delta w_{b,(t-1):t}^{(cor)} + \Delta w_{b,(t-1):t}^{(vol)} + \Delta w_{b,(t-1):t}^{(mean)} \quad (6)$$

where the last three terms are time differences of the three decomposed optimal weights. Later we find that the correlation coefficient between $\Delta w_{b,(t-1):t}^{(c+v)}$ and $\Delta w_{b,(t-1):t}^{(cor)} + \Delta w_{b,(t-1):t}^{(vol)}$ is 0.99, implying that our Bitcoin demand decomposition based on Equation (4) produces negligible approximation errors.

Our decomposition of Bitcoin demand proxy is built on a simple binary classification of investors. This classification is only conceptual rather than physical. For example, an investor might have both types of motives (speculative and non-speculative) for her investments. If so, our decomposed proxies captures the total demand for each motive, aggregated across investors.

3.3 Proxy Estimation and Diversification Benefits

Speculative investors only care about Bitcoin valuation based on their private information, subjective interpretation, and a Bitcoin pricing model of their own choice. We have no access to speculative investors' information sets or subjective belief. Therefore, constructing a demand proxy for speculative investors is challenging. Instead, we focus on the non-speculative Bitcoin demand in Equation (4) due to learning of correlations and volatilities.

We estimate (ρ_t, σ_t^*) using a multivariate GARCH model, DCC(1,1)-GARCH(1,1), as described in Section 2. Therefore, our approach of proxy estimation is based on two assumptions. First, we restrict non-speculative investors' information set to historical returns of Bitcoin and the market portfolio. This is the smallest possible information set to estimate (ρ_t, σ_t^*) ; this parsimonious approach is the most popular among both academics and practitioners. Furthermore, the approach is immune to manipulation; it prohibits us from cherry-picking certain information set. Second, we assume that non-speculative investors believe the multivariate GARCH model used in our proxy construction is a true data generating model. Yet, we find that the empirical results are practically invariant to a choice of time-series model.

By contrast, we assume that non-speculative investors do not learn about the conditional risk premium ratio μ_t^* .²⁴ Throughout this paper, we simply fix $\bar{\mu}^*$ at a specific value, $\bar{\mu}^* = 1$, to avoid

²⁴The signal-to-noise ratio for a mean parameter is generally large because of its statistical nature. Furthermore, since a mean parameter estimate usually follows a Normal distribution, the ratio of two mean parameter estimates follows a Cauchy distribution whose mean and any other moments are indefinite. These challenges also applies to non-speculative investors, let alone econometricians. Therefore, it is reasonable to assume that non-speculative investors do not learn about μ_t^* at daily frequency.

cherry-picking an arbitrary value.²⁵ Here, $\bar{\mu}^* = 1$ means that Bitcoin’s risk premium is as low as the market risk premium. Since Bitcoin has high volatility and its fundamental value is extremely uncertain, no investors will purchase Bitcoin when they believe Bitcoin has lower risk premium than the market portfolio. At $\bar{\mu}^* = 3.26$, Bitcoin and S&P500 have the same ex-ante Sharpe ratio for the post-futures period, given their sample volatilities in that period. Since the CAPM implies that the market portfolio has the highest Sharpe ratio, reasonable values for $\bar{\mu}^*$ should lie in this range $[0, 3.26]$. Furthermore, we suspect that passive investors are possibly cautious and conservative; therefore, their belief on $\bar{\mu}^*$ is in the lower side of the range, inspired by the literature in model misspecification, ambiguity or Knightian uncertainty.²⁶

Other unconditional estimates $(\bar{\sigma}^*, \bar{\rho})$ are fixed at their medians of (ρ_t, σ_t^*) from the multivariate GARCH estimation, respectively. Since we eventually use changes in demand as a main predictor rather than its level, our main results remain practically invariant to our choice on $(\bar{\mu}^*, \bar{\sigma}^*, \bar{\rho})$ values and multivariate GARCH models, as shown in Section A.8.1 in the online appendix. Figure 2 shows the decomposed *excess* non-speculative demands $w_{b,t}^{(cor)} - \bar{w}_b$ and $w_{b,t}^{(vol)} - \bar{w}_b$ in the post-futures sample. Summary statistics can be found in Table A.2 in the online appendix.

Diversification Benefits of Bitcoin Since late 2017, the correlation between Bitcoin and S&P500 is time-varying and overall positive. The daily volatility of Bitcoin in the post-futures period is five times larger than that of S&P500. However, non-speculative investors’ demand proxy $w_{b,t}^{(c+v)}$ proves that Bitcoin’s diversification benefits still exist. Table 3 shows that including Bitcoin in a passive portfolio (S&P500) improves its ex-ante daily Sharpe ratio at least 83.5% of the time, even with the most conservative belief on Bitcoin’s risk premium, i.e., $\bar{\mu}^* = 1$. The median increase in Sharpe ratio goes up to 14.5% when investors’ belief on Bitcoin’s risk premium is relatively high ($\bar{\mu}^* = 3$) but still reasonable. We provide additional results in the online appendix A.7.

4 Bitcoin Return Predictability at Full Scale

Using our decomposed demand proxies, we present a list of empirical results in Bitcoin return predictability: Granger causality, in-sample estimations with linear models, variable comparisons

²⁵We show later that our empirical results are not sensitive to $\bar{\mu}^*$.

²⁶See Hansen and Sargent (2008), Epstein and Schneider (2003), and Klibanoff et al. (2005).

with machine learning methods, out-of-sample predictions, and robustness tests. All results confirm that the Bitcoin return predictability truly exists when the correlation is time-varying.

4.1 Data

We download Bitcoin and S&P500 daily closing price data from *coinmarketcap.com* and *investing.com*, respectively.²⁷ Then, we combine multi-day Bitcoin returns to one-period return whenever the US stock market is not traded. We select the data from January 2015 to December 2020 (henceforth, ‘full sample’) since our focus is traditional investors rather than early adopters. As a result, the full sample includes 1,511 daily returns of Bitcoin and S&P500 index. Similarly, we obtain NYSE composite and NASDAQ composite indexes plus international equity indexes, ranked by nominal GDP as of 2019: Shanghai Composite (China), Nikkei 225 (Japan), BSE Sensex (Germany), FTSE 100 (India), DAX (UK), CAC 40 (France), and FTSE Italia All-Share (Italy) indexes.²⁸ We adjust the international data according to their daily foreign exchange rates from the futures markets.²⁹ Other cryptocurrency Ethereum (ETH) and Ripple (XRP) are downloaded from *coinmarketcap.com*. In addition, SKEW, VIX, and S&P500 trading volume data are acquired from *investing.com*.

To isolate the effect of learning, we collect other variables that are known to predict Bitcoin returns in literature (e.g., [Liu and Tsyvinski 2021](#)): USD index returns and gold returns from *investing.com*, daily treasury yield rates from the US Department of Treasury, blockchain-related attributes including total hashrates, global block difficulty, unique address count, total Bitcoin quantity, unique transactions count, and trading volume on major Bitcoin exchanges from *blockchain.com*, Wikipedia Bitcoin pageviews from Wikipedia *Pageviews Analysis*, and Google search trends for Bitcoin from *trends.google.com*. The Economic Policy Uncertainty Index data based on daily news are obtained from *policyuncertainty.com*. See Table 5 for the complete list of predictors we consider.

²⁷The US stock market closes at 4pm Eastern Standard Time (EST) whereas the Bitcoin market closes at 12am Coordinated Universal Time (UTC). Bitcoin closing price is recorded 3 or 4 hours later than the S&P500 index on a given day depending on daylight saving. For simplicity, we ignore this time difference and use closing price of both assets in our analyses.

²⁸International equity market have different daily closing times. We use forward market open index levels for China’s and Japan’s markets while using closing index levels for other equity markets, to mitigate non-synchronous data issue.

²⁹We use forex open exchange rates for Chinese Yuan, Japanese Yen, European Euro, and British Pound whereas forex close exchange rates for Indian Rupee to mitigate non-synchronous data issue.

4.2 In-Sample Predictability

4.2.1 Time-Dependency in Returns and Granger Causality Tests

To lay the groundwork, we first investigate how lagged Bitcoin and stock market returns predict Bitcoin returns. Each column of Table 4 Panel A shows OLS (ordinary least squares) estimates on $a_1^{(L)}$ and their Newey-West t-statistics of the following univariate predictive regression for the post-futures sample:

$$r_{b,t+1} = a_0 + a_1^{(L)} z_{t+1-L} + \varepsilon_{t+1},$$

where $r_{b,t+1}$ is daily Bitcoin log returns and z_{t+1-L} is a lagged independent variable: Bitcoin demand changes due to correlation $\Delta w_{b,(t-L):t}^{(cor)}$ (standardized), Bitcoin returns $r_{b,t}$, or S&P500 returns $r_{m,t}$ for $L = 1, \dots, 4$. We observe that the coefficient is large (0.51) and significant (t-statistics = 4.15) only for $\Delta w_{b,(t-1):t}^{(cor)}$, that is, the lagged Bitcoin demand change due to correlations with lag $L = 1$. The coefficient reduces to less than its half (0.21) at the second lag $L = 2$ but its size is still the second largest among all in Panel A. The decreasing pattern in coefficients is consistent with the sequential learning effect that gradually fades.

We conduct the Granger causality tests (Granger 1969) in multivariate time-series setting:

$$r_{b,t+1} = a_0 + \sum_{L=1}^4 a_1^{(L)} z_{t+1-L} + \varepsilon_{t+1},$$

Table 4 Panel B reports three different regression results. First, we include lagged Bitcoin demand changes due to correlations and lagged Bitcoin returns. Second, we add lagged S&P500 returns to the first case. Finally, we repeat the second case using the weighted least squares (WLS) with weights that equal the inverses of time-varying variance estimates from DCC(1,1)-GARCH(1,1). We confirm that the same pattern as the univariate regressions in Panel A appears in all cases of the Granger causality tests in Panel B. Lagged Bitcoin returns and lagged S&P500 returns fail to predict Bitcoin returns in any case.

4.2.2 Predictive Regressions

Inspired by the learning-rebalancing mechanism, we express Bitcoin’s demand-return relationship in the following predictive regression model:

$$r_{b,t+1} = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + b_2 \Delta w_{b,(t-1):t}^{(vol)} + Z_t \gamma + \varepsilon_{t+1}, \quad (7)$$

where b_1 and b_2 measure how subsequent Bitcoin price reacts to the previous Bitcoin demand changes due to learning of correlation and volatilities, respectively. We include a large set of control variables Z_t to mitigate endogeneity issues, if any, in the predictive regression. Note our main predictor $\Delta w_{b,(t-1):t}^{(cor)}$ is a time difference and almost serially-uncorrelated; the first-order autocorrelation is -0.06 for the post-futures sample (-0.01 before COVID-19). Thus, our predictive regression is free from the common bias in predictive regressions (Stambaugh 1999).

Table 6 shows that a one-standard-deviation increase in the daily Bitcoin-demand-change $\Delta w_{b,(t-1):t}^{(cor)}$ (due to learning about correlation) predicts about 0.51% higher Bitcoin returns the next day in the post-futures sample.³⁰ R^2 in a univariate regression is 1.15% at daily frequency, which is larger than $R^2 = 1.05\%$ by $\Delta \rho_t$. The coefficient b_1 estimates remains stable around 0.5 and statistically significant in various specifications: with control variables, with rank-based estimation, without the sample after the COVID-19 outbreak, and with different stock market proxies (Table 8).

In contrast, Bitcoin-demand-changes due to learning about volatilities fail to predict Bitcoin returns. The coefficient b_2 estimates are small (ranging from 0.03 to 0.06 in the post-futures period) and not statistically significant. R^2 is only 0.01% in a univariate regression. Furthermore, time-varying daily CAPM β_t , constructed by DCC-GARCH estimates, barely affects next-day Bitcoin returns. While Bianchi and Dickerson (2019) show that trading volume plays a role in predicting Bitcoin returns, we find that lagged trading volume of Bitcoin negatively predicts Bitcoin returns but becomes statistically insignificant when other Bitcoin attributes are included as controls in regression. Economic Policy Uncertainty (EPU) Index with a coefficient estimate around 0.63 is positively associated with subsequent Bitcoin returns but shy of statistical significance (t-statistics between 1.68 and 1.76). Above all, Bitcoin daily return predictability largely comes from investors’ learning about correlation rather than volatilities or other predictors concerned.

³⁰For this interpretation to make sense, we standardize $\Delta w_{b,(t-1):t}^{(cor)}$ and $\Delta w_{b,(t-1):t}^{(vol)}$.

4.2.3 Relative Importance of Predictors

Changes in Bitcoin demand due to time-varying correlations certainly predict subsequent daily Bitcoin returns. However, linear regression models might suffer from over-fitting and do not clearly show how strong the predictability evidence is relative to other commonly used predictors.

To address these concerns, we apply several Machine Learning algorithms to fit the data: LASSO, Group LASSO, Elastic Net, Random Forest (RF), Gradient Boosting Machine (GBM), and Neural Network with one hidden layer (NN1). In particular, the last three algorithms consider non-linearity and all possible interactions among predictors. Table 5 shows a list of all predictors simultaneously used in training. To avoid over-fitting, we perform a five-fold cross-validation to tune hyper-parameters for each algorithm by a grid search.³¹

To quantify the importance of each predictor, we compute a reduction in R^2 , MSE (Mean-Squared-Errors), or MAE (Mean-Absolute-Errors) when a given predictor is set to zero in a trained model, following Gu et al. (2020). We report only the results based on R^2 since all three measures produce very similar outcomes in importance metrics. Two heat maps in Figure 3 visualize this variable-importance metric for several machine learning algorithms. In Panel (a), the variable importance metric is scaled by the column-wise sum so that the total sum for each column can be unity. Darker colors represent greater importance in prediction. Panel (b) repeats the Panel (a) but scales the importance metric by normalizing the variable importance metric to be between zero and one for each algorithm.

Finally, Table 7 reports the in-sample importance metric numbers for the most and the second most important predictors for each algorithm using post-futures data. The most important variable (MIV) is dominated by $\Delta w_{b,t}^{(cor)}$ and lagged trading volume; $\Delta w_{b,t}^{(cor)}$ is the top predictor in Group LASSO (GLASSO), tree-based models (RF and GBM), and an artificial neural network (NN1) while lagged trading volume is only dominant in certain linear models, LASSO and Elastic Net. It is noticeable that $\Delta w_{b,t}^{(cor)}$ is always among the top two important predictors whereas all other predictors fall short.

³¹We tune the shrinkage factors in LASSO, Group LASSO, and Elastic Net. For Random Forest, we tune the feature sampling size and the minimum size of terminal nodes. For Gradient Boosting Machine, we tune the number of trees, the shrinkage factor, the maximum depth of trees, the number of minimum observation in each node, and the fraction of sub-sampling. For Neural Network, we tune the L1 shrinkage factor and the learning rate. For all algorithms, we use a mean absolute error loss function for performance evaluation.

From this Machine Learning exercise, we conclude that our proposed predictor is not only orthogonal to other predictors but also a dominant predictor for daily Bitcoin returns, especially when non-linearity and possible interactions are considered.

4.3 Out-of-Sample Predictability

We perform out-of-sample tests to address potential issues in the in-sample estimation: over-fitting, look-ahead biases, coefficient instability, and investors' learning on parameters in addition to correlations and volatilities. We re-estimate the DCC(1,1)-GARCH(1,1) and fit predictive regression models every trading day in real time to forecast sequentially Bitcoin returns using $\Delta w_{b,(t-1):t}^{(cor)}$ or $\Delta \rho_t$ without any look-ahead bias. We increase the estimation (training) window after initial training with one year of data such that our prediction period starts from the beginning of 2019. In Table 9, we report out-of-sample R^2 statistics, suggested by [Campbell and Thompson \(2008\)](#):

$$R_{OS}^2 = 1 - \frac{SSE(p)}{SSE(h)}, \quad (8)$$

where $SSE(p)$ is the sum of squared forecast errors by our real-time predictive regression model whereas $SSE(h)$ is by the historical average of the past Bitcoin returns as a benchmark forecast. Positive R_{OS}^2 is regarded as evidence for return-predictability beyond that by the benchmark forecast.

While R_{OS}^2 uses historical average returns as a benchmark, [Gu et al. \(2020\)](#) argue that this benchmark is flawed when analyzing individual stock returns with noisy historical averages. To deal with that, they propose a modified out-of-sample performance measure, denoted as R_{OS}^{2*} , that compares prediction against a forecast value of zero:

$$R_{OS}^{2*} = 1 - \frac{SSE(p)}{SSE(0)} \quad (9)$$

In other words, $SSE(0)$ in R_{OS}^{2*} uses flat zero as a benchmark forecast rather than historical average returns. In our test sample, R_{OS}^{2*} is lower than R_{OS}^2 , which means the historical average performs worse than *zero as a forecast* for Bitcoin returns.

In general, R_{OS}^2 is typically lower than in-sample R^2 . We find the same pattern in the ordinary

least-squares estimation (OLS) in Table 9. However, our robust alternatives (WLS, LAD and Rank regressions) even outperform the in-sample predictions. Even though R_{OS}^{2*} is generally lower than R_{OS}^2 , it largely outperforms in-sample R^2 for median and rank-based estimation (LAD and Rank). Therefore, concerns about potential over-fitting and coefficient instability are well cleared. Also, outperforming out-of-sample forecasts suggest that investors may sequentially learn about the parameters in the time-series model in addition to the time-varying correlations and volatilities. Strong out-of-sample predictability by robust linear regression models also suggests that our return predictability is not driven by outliers or influential points since such methods are less sensitive to the extreme observations.

We find that the predictability is stronger with globally aggregated Bitcoin demand weighted by GDP, producing $R_{OS}^{2*} = 1.74\%$ at daily frequency with LAD regressions.³² Excluding the COVID-19 period sample barely changes the predictability. This out-of-sample predictability evidence is impressive even to practitioners. [Campbell and Thompson \(2008\)](#) argue that even a small R_{OS}^2 , such as 0.5% at monthly frequency, can deliver economically meaningful return predictability to investors, let alone at daily frequency. Therefore, this out-of-sample Bitcoin return predictability offers a valuable trading opportunity to investors in practice.

4.4 Robustness Tests

We perform extensive robustness tests to solidify the return predictability evidence. First, we construct Bitcoin Demand proxy in many different ways to see the sensitivity of our results: using other types of multivariate GARCH models and fixing $(\mu_t^*, \sigma_t^*, \bar{\rho})$ at other values. Second, we run quantile regressions at different quantiles to confirm robustness against non-normality and extreme observations. Third, we trim or winsorize the data since Bitcoin returns are highly volatile and fat-tailed. Lastly, we repeat out-of-sample tests by computing a confusion matrix and performing related tests for a binary classifier (price up/down) rather than regressions. We find that our return predictability results are highly robust and practically invariant to all these alternative specifications. We provide the detailed results in the online appendix [A.8](#).

³²We align observations from global aggregate predictors with those from S&P500-based predictor such that predictability between S&P500-based and global-based predictors can be compared.

5 Testable Predictions of Learning-Rebalancing Explanation

This section investigates five groups of testable predictions implied by our explanation on Bitcoin return predictability, and we find that all predictions are consistent with empirical evidence provided here, which is highly unlikely to happen all by chance. The results in this section also mean that our decomposed demand proxies can uncover relevant empirical evidence that conditional correlation estimates alone cannot reveal.

5.1 Prediction #1: Global Equity and Cryptocurrency Markets

Bitcoin is not limited to the US investors. Any international investors who optimize their portfolios using Bitcoin and equity markets in their own countries contribute to Bitcoin demand. This global aspect of Bitcoin provides two testable predictions: (1) Bitcoin’s return predictability should appear also with international equity markets if passive investors around the globe behave in the same way as those in the US, and (2) if such learning and rebalancing affect Bitcoin prices through Bitcoin demand, return predictability should appear stronger when the Bitcoin demand proxy is constructed from the aggregated global Bitcoin demand instead of the US demand only. This second prediction differentiates the first prediction from a statistical artifact of correlated global equity markets. On the other hand, Bitcoin investors often include other cryptocurrencies in their portfolios for diversification within the cryptocurrency class. Hence, Bitcoin demand is naturally correlated with other cryptocurrency demands. If our demand proxy truly captures such a common component among cryptocurrency demands rather than idiosyncratic ones, our Bitcoin demand proxy should also predict other cryptocurrency returns.

Global Equity Markets We test the first prediction by repeating our analysis in Section 4.2.2 with seven major global equity markets ranked by GDP in 2019: Shanghai Composite (China), Nikkei 225 (Japan), BSE Sensex (Germany), FTSE 100 (India), DAX (UK), CAC 40 (France), and FTSE Italia All-Share (Italy) indexes. International evidence turns out comparable to the domestic one. In Table 8, the median R^2 of the eight countries (including the US) is 1.04% which is slightly lower than 1.15% from S&P500. The median coefficient is 0.49, close to 0.51 from S&P500. The median Newey-West t-statistics is 2.70, smaller than 4.15 from S&P500 possibly

due to higher volatilities in global equity markets. International evidence here confirms that our learning-rebalancing mechanism is a universal phenomenon, not a lucky outcome only in the US.

Globally Aggregated Demand Our analysis also confirms the second prediction. We aggregate the total Bitcoin demand changes of above eight countries using four different weighting schemes, as explained in Table 8. Note aggregating Bitcoin demand proxies across countries is conceptually more natural than aggregating correlation. The globally aggregated proxy for demand change produces overall higher coefficients (0.53 to 0.62) vs. 0.51 for S&P500, similar t-statistics (3.55 to 4.91) vs. 4.15 for S&P500, and higher R^2 (1.25% to 1.71%) vs. 1.15% for S&P500.³³ The superior prediction performance of globally aggregated demand proxies proves that the evidence in global equity markets for the first prediction is not a mere artifact of correlated equity markets. Furthermore, we find that the performance is stronger when the demand is aggregated with economically sensible weights such as GDP. Also, the globally aggregated demand proxy with GDP outperforms a combination forecast (equal-weighted average of forecasts from individual regressions) implying that this global aggregation is beyond a statistical trick.

Other Cryptocurrencies To test the prediction on other cryptocurrencies, we repeat the predictive regression analysis and find similar predictability evidence while replacing the dependent variable (i.e., subsequent daily Bitcoin returns) by Ethereum (ETH) or Ripple (XRP) returns.³⁴ All these results suggest that our demand proxy predicts the common components in the cryptocurrency market rather than idiosyncratic parts.³⁵

5.2 Prediction #2: Expected Errors in Optimal Portfolio Weights

Section 4.2.2 shows that changes in Bitcoin demand due to learning about volatilities fail to predict Bitcoin returns (See $\Delta w_{b,t}^{(vol)}$ in Table 6). This empirical result implies that investors do not learn about volatilities for portfolio optimization even if our learning-rebalancing mechanism truly

³³Given that countries have different holidays, at each date a combined predictor is calculated according to available entries and only skipped if no equity markets are open at that date.

³⁴We do not report here due to space limits but Litecoin shows similar results. We exclude Tether because it is a stable coin that lacks uncertainty in our mechanism.

³⁵We add extensive robustness tests related to Table 8. For example, we find similar significant results when we construct a cryptocurrency demand proxy using Ethereum and Ripple instead of Bitcoin.

generates the Bitcoin predictability. If so, it must be because investors' volatility forecasts are too noisy to be inputs for portfolio optimization.³⁶

Investors understand that the calculated portfolio weights are only ex-ante optimal, not ex-post. Even if investors are doing their best in learning time-varying correlations and volatilities, investors' estimates on them can be inaccurate the next day when such hidden state variables dramatically change overnight. Such fundamental tracking errors in optimal portfolio add extra volatilities to their portfolios. In fact, this is why a naïve "1/N" equal-weight portfolio often outperforms a portfolio with optimized weights. Therefore, rational investors who understand the impact of this tracking errors would suppress or even ignore noisy rebalancing signals.

We test this prediction by comparing the degrees of time-variation in correlation, volatilities, and optimal portfolio weights. The first two columns in Table 10 show that the estimated volatility ratio σ_t^* is less persistent and much more volatile than the estimated correlations ρ_t . The fitted AR(1) models imply that investors face 3.48 times greater errors in volatility ratio than in correlation, in terms of the forecasting mean-squared-error. Such a difference in errors explodes when it transfers to corresponding errors in optimal portfolio weights. Investors face 30 times greater expected errors in $w_{b,t}^{(vol)}$ than in $w_{b,t}^{(cor)}$, as shown in the last two columns in the table. Given this dramatic difference in expected errors, rational investors would direct all their attention to correlation changes while ignoring changes in volatility ratio. This analysis also confirms that constructing a demand proxy is the key to understanding Bitcoin return predictability.

5.3 Prediction #3: Bitcoin Demand Proxy vs. Correlation Estimates

If passive investor's learning and rebalancing affect Bitcoin prices through Bitcoin demand, the return predictability should be stronger when the predictor is constructed from the demand proxy rather than correlation estimates. This prediction is consistent with our analysis in Section 4.2.2, but the evidence is weak because two predictors are highly (negatively) correlated. In this section, we refine the Bitcoin demand proxy and provide stronger empirical evidence consistent with the prediction.

³⁶It is well known that return volatilities are time-varying. Therefore we ignore the case in which investors falsely believe volatilities are constant.

A Conditional Proxy for Bitcoin Demand Changes Section 3.2 defines a proxy for Bitcoin demand changes as follows: $\Delta w_{b,(t-1):t}^{(cor)} = w_{b,t}^{(cor)} - w_{b,t-1}^{(cor)}$ where $w_{b,t}^{(cor)} = w_{b,t}(\bar{\mu}^*, \bar{\sigma}^*, \rho_t)$. Therefore, this proxy is independent of the time-variation of volatility ratio σ_t^* . Thanks to such an unconditional aspect, the resulting proxy is conservative and reliable but loses conditional information in volatility ratio σ_t^* . Investors would ignore a change in volatility ratio because of its noisiness, yet they can still utilize the level of volatility ratio as an input for optimal portfolio weight because estimates on levels are less noisy than changes in such estimates. Hence, we newly define a conditional version of a proxy for Bitcoin demand changes as follows.

$$\Delta w_{b,(t-1):t}^{(cor|vol)} = w_{b,t}(\bar{\mu}^*, \sigma_{t-1}^*, \rho_t) - w_{b,t-1}(\bar{\mu}^*, \sigma_{t-1}^*, \rho_{t-1}) \quad (10)$$

Note the level of volatility ratio is fixed at σ_{t-1}^* at both $t - 1$ and t . Therefore, investors ignore changes in volatility ratio as confirmed in Section 5.2 and do not revise the optimal weight because of potential changes in volatility ratio. However, the overall level of volatility ratio affects the calculations of optimal portfolio weights. Given the same level of correlation, the optimal portfolio weight depends on the level of volatility ratio.³⁷

In-Sample Predictability Comparison Table 11 compares Bitcoin return predictability by three different predictors. The first column shows the result by changes in correlation $\Delta\rho_t$ from Table 2. The second and third columns show the results by original unconditional proxy $\Delta w_{b,(t-1):t}^{(cor)}$ in Equation (5) from Table 6 and by the new conditional proxy $\Delta w_{b,(t-1):t}^{(cor|vol)}$ in Equation (10), respectively. As expected by the learning-rebalancing channel, the R^2 's of three univariate regressions are in the right sequence: 1.05% (changes in correlations), 1.15% (unconditional proxy), and 1.59% (conditional proxy). The next four columns show that the conditional proxy makes other two predictors insignificant when they are included together in multivariate regressions. The last column shows the Pearson correlation coefficients between three predictors. The original unconditional proxy is highly negatively correlated with changes in correlations, $Corr(\Delta\rho_t, \Delta w_{b,(t-1):t}^{(cor)}) = -0.99$. Therefore, they do not differ much in return predictability. However, the new conditional proxy is

³⁷This new proxy is still an approximation. Investors might use a moving average of volatility ratios in the past few days instead. However, we find that the empirical results remain qualitatively similar regardless of how we approximate the overall level of volatility ratio.

less correlated with changes in correlations, $Corr(\Delta\rho_t, \Delta w_{b,(t-1):t}^{(cor|vol)}) = -0.93$, so, return predictability can be greatly improved.

Out-of-Sample Predictability Comparison Table 12 shows out-of-sample return prediction performances by three predictors: $\Delta\rho_t$, $\Delta w_{b,(t-1):t}^{(cor)}$, and $\Delta w_{b,(t-1):t}^{(cor|vol)}$. We repeat Table 9 using two robust linear models: least absolute deviation (LAD) and Rank regressions. Consistent with the in-sample evidence in Table 11, the R^2 's of univariate predictive regressions remain in the right order: $\Delta\rho_t$ generates the smallest R^2 while $\Delta w_{b,(t-1):t}^{(cor|vol)}$ produces the largest. This pattern is universal whether LAD or Rank is used and whether the proxies are constructed from the US equity markets (S&P500) or globally aggregated equity markets. Out-of-sample R^2 is maximized by Rank regression with the globally aggregated conditional proxy, up to $R_{OS}^2 = 3.40\%$ and $R_{OS}^{2*} = 2.29\%$ at daily frequency.³⁸

5.4 Prediction #4: Conditional Predictability of Bitcoin Demand Proxy

The results in Section 5.3 imply that the prediction performance of $\Delta\rho_t$ and $\Delta w_{b,(t-1):t}^{(cor)}$ are conditional on volatility ratio. Those variables predict Bitcoin returns better at certain levels of volatility ratio but worse at other levels. If so, the pattern of such conditional predictability should be consistent with the pattern of Bitcoin demand implied by passive investor's learning-rebalancing mechanism. First, using Bitcoin demand proxies, we first show that the learning-rebalancing mechanism theoretically implies relatively weaker predictability when volatility ratio is too high or too low. Second, the Bitcoin demand proxy should be more informative and result in stronger return predictability when the proxy is constructed with reasonable values of risk premium ratio $\bar{\mu}^*$. We find that both predictions are consistent with empirical evidence.

5.4.1 Conditional on Volatility Ratio

Figure 4 Panel (a) shows the impact of correlation changes on Bitcoin demand at different volatility ratio levels. We compute the time-series of $\Delta w_{b,(t-1):t}^{(cor)}$ and its sample standard deviation with $\bar{\sigma}^*$ fixed at each σ_t^* for $t = 1, 2, \dots, T$. Panel (a) shows the sample standard deviation for each choice of $\bar{\sigma}^*$ in the horizontal axis, expressed as its empirical quantile in the set $\{\sigma_t^*\}_{t=1}^T$. The sample

³⁸See Section 4.3 for the difference between R_{OS}^2 and R_{OS}^{2*} .

standard deviation of $\Delta w_{b,(t-1):t}^{(cor)}$ measures the sensitivity of Bitcoin demand changes to correlation changes. Therefore, return predictability should be weak when the standard deviation is small, for example, when volatility ratio is extremely high or low, as shown in In Panel (a). If so, removing such observations in a predictive regression should improve return predictability.

Figure 4 Panel (b) confirms this prediction empirically. We gradually trim the observations with extremely high or low volatility ratio based on the graph in Panel (a). For example, we first pick a threshold of the standard deviation in the vertical axis of Panel (a). Then remove observations with lower standard deviation than the threshold. As a result, only the observations in the middle of Panel (a) remain for a predictive regression. We repeat this procedure for higher thresholds to remove more observations gradually. Panel (b) reports the Newey-West t-statistics of three different predictors $\Delta \rho_t$, $\Delta w_{b,(t-1):t}^{(cor)}$, and $\Delta w_{b,(t-1):t}^{(cor|vol)}$, from univariate predictive regressions with different trimming schemes, i.e., different thresholds.³⁹ As observations with an extreme volatility ratio are trimmed, the t-statistic and return predictability increase in all three predictors.⁴⁰

The degree of improvement, however, differs in three predictors. For the conditional proxy $\Delta w_{b,(t-1):t}^{(cor|vol)}$, the improvement is almost negligible, consistent with our prediction. By construction, the conditional proxy already incorporates the information, i.e., conditional predictability, visualized in Panel (a). Therefore, we should not observe a further improvement in return predictability by removing observations with weak predictability. By contrast, the other predictors $\Delta \rho_t$ and $\Delta w_{b,(t-1):t}^{(cor)}$ show a large improvement in predictability. In particular, the t-statistic of the unconditional proxy $\Delta w_{b,(t-1):t}^{(cor)}$ increases and eventually converges to that of the conditional proxy $\Delta w_{b,(t-1):t}^{(cor|vol)}$. Therefore, the superior predictability of the conditional proxy in Section 5.3 is not a coincidence but truly driven by the conditional predictability, as shown in Figure 4 Panel (a).

5.4.2 Conditional on Risk Premium Ratio

Another variable in the optimal portfolio weight is risk premium ratio $\bar{\mu}^*$. Since passive investors belief on $\bar{\mu}^*$ is unobservable and difficult to estimate, we first suggest its reasonable values between 1 and 3.26 in Section 3.3. Yet, passive investors are relatively more cautious and conservative rather

³⁹Here, R^2 is not a proper measure for return predictability comparisons, because removing observations alters baseline R^2 .

⁴⁰Note trimming here is based on extreme volatility ratios, not on extreme returns. Therefore, the improvement in return predictability is not a result of removing outliers (extreme returns).

than aggressive; therefore, their belief on $\bar{\mu}^*$ is arguably in the lower side of the range. Therefore, the Bitcoin demand proxy should be more informative and show stronger return predictability when the proxy is constructed with reasonable values of risk premium ratio $\bar{\mu}^*$, in particular, in the lower half of the range. We find empirical evidence consistent with this prediction.

Figure 5 shows how return predictability varies depending on the value of $\bar{\mu}^*$ used in the Bitcoin demand proxy construction. As expected, the conditional proxy $\Delta w_{b,(t-1):t}^{(cor|vol)}$ outperforms the unconditional proxy $\Delta w_{b,(t-1):t}^{(cor)}$ only when $\bar{\mu}^*$ is fixed roughly between 1 and 2, in terms of Newey West t-statistics in Panel (a) and R^2 in Panel (b). The conditional proxy $\Delta w_{b,(t-1):t}^{(cor|vol)}$ extracts information on Bitcoin demand more aggressively. Therefore, the conditional proxy becomes too noisy when $\bar{\mu}^*$ is fixed at unreasonable values. By contrast, the original unconditional proxy $\Delta w_{b,(t-1):t}^{(cor)}$ is less efficient in capturing information on Bitcoin demand but more robust to the use of extreme values of $\bar{\mu}^*$ in proxy construction. Based on these results, we argue that the passive investors belief on $\bar{\mu}^*$ is arguably between 1 and 2, consistent with the discussion in Section 3.3.

5.5 Prediction #5: Placebo Tests

If the Bitcoin predictability truly arises from our learning-rebalancing mechanism rather than from a statistical artifact, the following placebo tests should be rejected. We confirm such predictions.

5.5.1 Return Predictability with Constant Correlation?

Prior to the Bitcoin futures markets, investors perceive risk in Bitcoin as purely idiosyncratic, and the conditional correlation between Bitcoin and the stock markets is practically fixed at zero (Figure 1 and Table 1). If the predictability comes from time-varying correlation and investors' learning on it, we should observe absence of predictability when the correlation is practically constant and incentive of learning is relatively small.

To test this idea, we generate illusory time-varying correlations (the gray region in Figure 1) for the pre-futures sample, by using the DCC-GARCH estimates from the full-sample (or the post-futures sample). This process creates an illusion that allows us to perform a placebo test; Bitcoin return predictability should disappear for the pre-futures sample. It is because investors do not recognize these virtual-correlation-based signals for rebalancing in real time prior to the Bitcoin futures markets (Figure A.2 in the online appendix). As expected, return predictability disappear

in the pre-futures period when the correlation is constant. See Table A.5 in the online appendix for a detailed analysis.

5.5.2 Return Predictability in Stock Markets?

We argue that return predictability in stocks is difficult to find even if we apply the same approach in Section 3 to the stock market returns instead of Bitcoin. First, most of Bitcoin investors also invest in stocks, but only a fraction of stock investors consider Bitcoin seriously and learn their joint dynamics. Thus, the learning-rebalancing channel is less important in the demand for stocks. Second, fluctuating weights on the Bitcoin-stock portfolio affect asset demand asymmetrically because of their market capitalization difference. For example, when the weight on Bitcoin jumps eight times from 0.3% to 2.4%, the weight on stocks only drops from 99.7% to 97.6%. Therefore, the effect of learning should be negligible in stocks, compared to Bitcoin. As expected, we do not find evidence for stock return predictability using our demand proxy. In a univariate predictive regression for S&P500 index returns, the coefficient on $\Delta w_{b,(t-1):t}^{(cor)}$ is 0.11 while its t-statistic is 1.07.

6 A Static Model of Bitcoin Prices

This section develops a static model of equilibrium Bitcoin prices by formalizing our intuitive explanation in Section 3.1. The model can quantitatively explain Bitcoin’s return predictability puzzle in Table 2 by producing the observed coefficients in the predictive regressions with the changes and levels of correlation as a predictor.

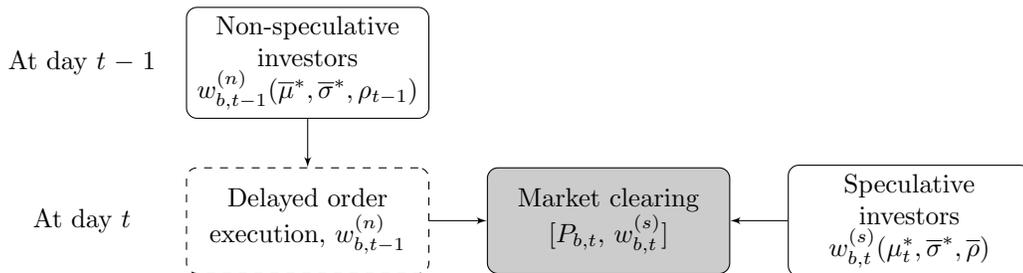
6.1 Bitcoin Demands and Timeline

Following Section 3, we split investors, or investors’ motives, into the speculative and the non-speculative. Then we interpret both investors’ trading decisions in a mean-variance framework to construct Bitcoin demands, similarly to Section 3.2. If an investor trades Bitcoin for price speculation, she would barely care about diversification. The investor rather directs all her limited attention and resources to learn about the risk premium. If an investor manages a passive portfolio, she believes that predicting highly volatile Bitcoin’s price is a difficult task and only adds unnecessary noises to the portfolio optimization. Therefore, she would focus on learning about

correlations rather than the risk premium.⁴¹

The optimal weights on Bitcoin in both types of investors' portfolios are determined by Equation (2) with their own subjective input values, respectively. For non-speculative investors, the optimal weight is denoted by $w_{b,t-1}^{(n)}$ which equals $w_{b,t-1}^{(cor)} = w_{b,t-1}(\bar{\mu}^*, \bar{\sigma}^*, \rho_{t-1})$. In other words, non-speculative investors care about only time-varying correlations. For speculative investors, the optimal weight on Bitcoin at day t is denoted by $w_{b,t}^{(s)}$ which equals $w_{b,t}^{(mean)} = w_{b,t}(\mu_t^*, \bar{\sigma}^*, \bar{\rho})$. That is, speculative investors concern about only the risk premium of assets, or equivalently, only the current asset prices given the expected future prices. Therefore, even if speculative investors in the real world do not maximize Sharpe ratio, we can still rely on the mean-variance framework to explain their speculative investment decisions simply by adjusting μ_t^* accordingly.⁴²

Unlike the approach in Section 3, the evolution of investors beliefs is disregarded in the equilibrium model. Technically, this is not a dynamic learning model but a static model. The model starts with a snapshot of investors' beliefs at day $t - 1$ as exogenous variables and then completes an equilibrium at day t . The following flowchart visualizes how Bitcoin's market clearing occurs by the two types of investors, as a high-level summary.



The key feature of the model comes from the intuition in Section 3.1: *asynchronous portfolio rebalancing*. A non-speculative fund manager revises her estimate on time-varying correlation at market close. Then the fund manager requests traders to rebalance her portfolio with new optimal weights. Here, the fund manager does not necessarily rebalance every day. Different funds can be rebalanced on different days while they rebalance only once in a while, e.g., once a month.⁴³ Then,

⁴¹Section 5.2 explains why Bitcoin investors are likely to ignore time-variation in volatilities.

⁴²The distinction between two types of investors are not physical. One investors can have both speculative and non-speculative motives, and we can interpret her Bitcoin demand as a sum of the demands of two different agents inside her.

⁴³See discussion in Section 6.5 for the validity of assumption on daily rebalancing.

instead of executing orders immediately, the traders in practice split and delay orders during the next trading session while hoping for better execution prices without revealing the fund manager’s intentions.⁴⁴ Bitcoin price at day t does not affect non-speculative investors’ Bitcoin demand, in terms of portfolio weight $w_{b,t-1}^{(n)}$, which is already set on the previous day.⁴⁵

Unlike our empirical proxy construction, our equilibrium model puts speculative investors into play to induce Bitcoin’s market clearing in an equilibrium. Speculative investors’ demand, by contrast, is concurrent as they instantly react to the real-time prices. Furthermore, the Bitcoin price is directly linked to speculative investors’ subjective Bitcoin risk premium $\bar{\mu}^*$ and thus to their Bitcoin demand $w_{b,t}^{(s)}$. Instead of calibrating their subjective Bitcoin risk premium $\bar{\mu}^*$, we exogenously set speculative investors’ subjective expectation on future Bitcoin prices.

6.2 Definitions and Calibration of Variables

First, let $Q_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(n)}$ denote Bitcoin quantity held by speculative and non-speculative investors at day $t-1$, respectively. Second, to keep notation simple, we normalize both the total Bitcoin quantity and Bitcoin price at day $t-1$ to unity, i.e., $P_{b,t-1} = 1$ and $Q_{b,t-1} = Q_{b,t-1}^{(s)} + Q_{b,t-1}^{(n)} = 1$, respectively. We also set the daily riskfree rates as zero, that is, $R_{f,t-1} = R_{f,t} = 1$ as a gross return.

We calibrate the other exogenous variables in the model, following Section 3.2. Two unconditional estimates $(\bar{\rho}, \bar{\sigma}^*)$ are fixed at (0.0946, 4.993%) which are their medians of (ρ_t, σ_t^*) , from the multivariate GARCH estimation, respectively. For non-speculative investors, we set $\bar{\mu}^* = 1$ and $\rho_{t-2} = \bar{\rho}$, and the optimal weight $w_{b,t-2}^{(n)} = 2.11\%$ is then determined by $(\bar{\mu}^*, \bar{\sigma}^*, \rho_{t-2})$ through Equation (2).⁴⁶ Therefore, non-speculative investors’ portfolio value is $A_{t-1}^{(n)} = Q_{b,t-1}^{(n)}/w_{b,t-2}^{(n)}$ since $P_{b,t-1} = 1$. Likewise, the optimal weight $w_{b,t-1}^{(n)}$ is determined by $(\bar{\mu}^*, \bar{\sigma}^*, \rho_{t-1})$.

For speculative investors, we express their portfolio value at day $t-1$ as $A_{t-1}^{(s)} = Q_{b,t-1}^{(s)}/w_{b,t-1}^{(s)}$ in which timing is slightly different from non-speculative investors’ case. Then we back out μ_{t-1}^* from Equation (2), given the values of exogenous variables $w_{b,t-1}^{(s)}$ and $(\bar{\sigma}^*, \bar{\rho})$. By setting speculative investors’ belief on daily market risk premium as $\mu_{m,t-1} = 0.06/252$, we can compute their Bitcoin

⁴⁴See Kyle (1985), Admati and Pfleiderer (1989), and an article related to this Bitcoin trading behavior at <https://yhoo.it/3i9ctPO>.

⁴⁵Therefore, their optimal weight on Bitcoin at day t is $w_{b,t-1}^{(n)}$ rather than $w_{b,t}^{(n)}$, and this is not an error in subscripts.

⁴⁶See a WSJ news article at <https://on.wsj.com/37clZgh>. Workers at participating companies can invest up to 5% of their account balances in cryptocurrency.

price forecast $E[P_{b,t}|\mathcal{F}_{t-1}^{(s)}]$ from the definition of μ_{t-1}^* :

$$\mu_{t-1}^* = \frac{\mu_{b,t-1}}{\mu_{m,t-1}} = \left(\frac{E[P_{b,t}|\mathcal{F}_{t-1}^{(s)}]}{P_{b,t-1}} - R_{f,t-1} \right) / \left(E[R_{m,t}|\mathcal{F}_{t-1}^{(s)}] - R_{f,t-1} \right). \quad (11)$$

Next, to shut down other channels that affect Bitcoin prices, we set the realized market portfolio gross return as $R_{m,t} = 1$ and assume that Bitcoin quantity does not change overnight, i.e., $Q_{b,t} = Q_{b,t-1} = 1$. Also, speculative investors' belief on market risk premium and future Bitcoin price remains unchanged from $t - 1$ to t , that is, $\mu_{m,t} = \mu_{m,t-1}$ and $E[P_{b,t+1}|\mathcal{F}_t^{(s)}] = E[P_{b,t}|\mathcal{F}_{t-1}^{(s)}]$. The following table organizes main variables in the model.

Day	Non-speculative	Speculative	Markets	
$t - 1$	ρ_{t-1}	$w_{b,t-1}^{(s)}, Q_{b,t-1}^{(s)}$		} Exogenous
	$Q_{b,t-1}^{(n)} = 1 - Q_{b,t-1}^{(s)}$			
	$A_{t-1}^{(n)} = Q_{b,t-1}^{(n)}/w_{b,t-2}^{(n)}$	$A_{t-1}^{(s)} = Q_{b,t-1}^{(s)}/w_{b,t-1}^{(s)}$		
	$w_{b,t-1}^{(n)}$	$E[P_{b,t} \mathcal{F}_{t-1}^{(s)}]$		
t		$E[P_{b,t+1} \mathcal{F}_t^{(s)}]$	$R_{m,t} = 1, Q_{b,t} = 1$	
	$Q_{b,t}^{(n)} = 1 - Q_{b,t}^{(s)}$	$\mu_t^*, w_{b,t}^{(s)}, Q_{b,t}^{(s)}$	$P_{b,t}$	} Endogenous

Given the variables in the first row, we can compute the other variables at day $t - 1$ and also those in the first row at day t . Therefore, the first five rows show exogenous variables outside an equilibrium since this model explains the equilibrium Bitcoin price only at day t .

6.3 A Static Equilibrium

Bitcoin price $P_{b,t}$ is endogenously determined at day t where two types of investors trade with each other as follows. Given a market price $P_{b,t}$ of Bitcoin, μ_t^* is computed by the day- t version of Equation (11). Then μ_t^* determines speculative investors' weight $w_{b,t}^{(s)}$ via Equation (2). Both types of investors' Bitcoin demands, $Q_{b,t}^{(s)}$ and $Q_{b,t}^{(n)}$, in terms of quantity, are set by Bitcoin demands, $w_{b,t}^{(s)}$ and $w_{b,t}^{(n)}$, in terms of portfolio weights, given Bitcoin price $P_{b,t}$. Therefore, Bitcoin price $P_{b,t}$

should result in market clearing at the equilibrium:

$$1 = Q_{b,t} = Q_{b,t}^{(s)} + Q_{b,t}^{(n)} = \frac{w_{b,t}^{(s)} A_t^{(s)}}{P_{b,t}} + \frac{w_{b,t-1}^{(n)} A_{t-1}^{(n)}}{P_{b,t}}, \quad (12)$$

where the timing difference between two types of investors comes from non-speculative investors' *asynchronous portfolio rebalancing*. Given the Bitcoin price $P_{b,t}$ and weight on Bitcoin $w_{b,t-1}^{(s)}$, the value of the speculative investor' portfolio evolves as follows.

$$A_t^{(s)} = A_{t-1}^{(s)} \left(w_{b,t-1}^{(s)} \frac{P_{b,t}}{P_{b,t-1}} + (1 - w_{b,t-1}^{(s)}) R_{m,t} \right), \quad (13)$$

By plugging Equation (13) into (12), we rewrite the market clearing condition (12) in the following.

$$\begin{aligned} P_{b,t} &= w_{b,t}^{(s)} A_t^{(s)} + w_{b,t-1}^{(n)} A_{t-1}^{(n)} \\ &= w_{b,t}^{(s)} A_{t-1}^{(s)} \left(w_{b,t-1}^{(s)} \frac{P_{b,t}}{P_{b,t-1}} + (1 - w_{b,t-1}^{(s)}) R_{m,t} \right) + w_{b,t-1}^{(n)} A_{t-1}^{(n)}, \end{aligned} \quad (14)$$

where we set $P_{b,t-1} = 1$ and $R_{m,t} = 1$. We numerically find the equilibrium Bitcoin price $P_{b,t}$ that satisfies the market clearing condition (14).⁴⁷

6.4 Model Implications on Predictive Regressions

Table 13 shows the model-implied coefficient $a_1^{(c)}$ in predictive regressions of Bitcoin returns $r_{b,t+1}$ on $(\rho_t - \rho_{t-1})$, i.e., changes in correlation between Bitcoin and the stock market returns:

$$r_{b,t+1} = a_0^{(c)} + a_1^{(c)}(\rho_t - \rho_{t-1}) + \varepsilon_{t+1} \quad \text{where} \quad a_1^{(c)} = \frac{1}{P_{b,t}} E \left[\frac{\partial P_{b,t+1}}{\partial \rho_t} \right] = E \left[\frac{\partial \log P_{b,t+1}}{\partial \rho_t} \right],$$

where we deviate ρ_{t-1} from $\rho_{t-2} = \bar{\rho}$ holding others constant. In Table 13, we compute $a_1^{(c)}$ using a numerical derivative for each combination of $w_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(s)}$ ranging from 0.1 to 0.9. The model-implied coefficient $a_1^{(c)}$ is negative regardless of $w_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(s)}$ as expected. The pattern in its magnitude is straightforward. Given $w_{b,t-1}^{(s)}$, predictability (or magnitude of the coefficient) decreases with speculative investors' share in Bitcoin, $Q_{b,t-1}^{(s)}$. High $Q_{b,t-1}^{(s)}$ means that non-speculative investors are only minor players in Bitcoin markets. Therefore, their demand

⁴⁷Alternatively, we can plug the $w_{b,t}^{(s)}$ expression into Equation (14) and solve a polynomial equation by brute force.

changes barely affect Bitcoin prices. On the other hand, Given $Q_{b,t-1}^{(s)}$, predictability increases with speculative investors' portfolio concentration in Bitcoin, $w_{b,t-1}^{(s)}$. Low $w_{b,t-1}^{(s)}$ means that speculative investors see relatively low expected returns on Bitcoin. Therefore, small percentage changes in the price can affect more dramatic percentage changes in Bitcoin's expected return and thus in their Bitcoin demand if $w_{b,t-1}^{(s)}$ is low. In other words, speculative investors are willing to absorb non-speculative investors' demand changes even with small price adjustments. Such a behavior of speculative investors leads to relatively weak predictability evidence when $w_{b,t-1}^{(s)}$ is low.

This calibration exercise tests whether the model can replicate the coefficient $a_1^{(c)}$ similar to its empirical estimate. The point estimate on $a_1^{(c)}$ of the regression using our data is -0.153 where its 95% confidence interval implied by Newey-West standard errors is $(-0.236, -0.072)$ (Table 2). Therefore, our model can easily produce the empirical estimate on $a_1^{(c)}$ with reasonable parameter values. For example, calibrating $w_{b,t-1}^{(s)} = 0.6$ and $Q_{b,t-1}^{(s)} = Q_{b,t-1}^{(s)}/Q_{b,t-1} = 0.3$ produces $a_1^{(c)} = -0.154$. This exercise does not prove that our intuition is an absolute truth, yet it provides a good indirect test for whether the intuition is completely off. Conversely, if this model is true, then we can estimate $w_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(s)} = Q_{b,t-1}^{(s)}/Q_{b,t-1}$ from the data although additional variables are possibly required for complete parameter identification.

We repeat Table 13 by comparing expected Bitcoin returns by different correlation levels, rather than its changes. Table 14 shows the model-implied coefficient $a_1^{(l)}$ in predictive regressions of Bitcoin returns $r_{b,t+1}$ on ρ_t , i.e., the level of correlation between Bitcoin and the stock market returns:

$$r_{b,t+1} = a_0^{(l)} + a_1^{(l)} \rho_t + \varepsilon_{t+1} \quad \text{where} \quad a_1^{(l)} = \frac{E[r_{b,t+1}^{(+)}] - E[r_{b,t+1}^{(-)}]}{\rho_t^{(+)} - \rho_t^{(-)}},$$

where $\rho_t^{(+)} = \bar{\rho}^* + SD(\rho_t)$, $\rho_t^{(-)} = \bar{\rho}^* - SD(\rho_t)$, $(\bar{\rho}^*, \rho_t)$ are the unconditional and conditional correlations from the DCC-GARCH estimation, respectively, and $SD(\rho_t) \approx 0.15$. On the other hand, $E[r_{b,t+1}^{(+)}]$ and $E[r_{b,t+1}^{(-)}]$ are the model-implied Bitcoin log returns corresponding to the correlation level $\rho_t^{(+)}$ and $\rho_t^{(-)}$, respectively. To compute $E[r_{b,t+1}^{(+)}]$, we first find $E[\rho_{t+1}|\rho_t^{(+)}]$ from Equation (1). Then compute the equilibrium Bitcoin prices at $t+1$ when $\rho_t = E[\rho_{t+1}|\rho_t^{(+)}]$ and $\rho_t = \rho_t^{(-)}$, respectively, holding others constant. Finally, we have a gross return that equals the ratio of those two prices, and its logarithm is $E[r_{b,t+1}^{(+)}]$. We compute $E[r_{b,t+1}^{(-)}]$ in a similar fashion.

The point estimate on $a_1^{(l)}$ of the regression using our data is practically zero, ($\hat{a}_1^{(l)} = 0.0006$),

where its 95% confidence interval implied by Newey-West standard errors is $(-0.0946, +0.0957)$. 80.2% of the model-implied coefficient $a_1^{(l)}$ in Table 14 fall between zero and 0.001. Most importantly, our model can simultaneously generate both $a_1^{(c)}$ and $a_1^{(l)}$ matching their empirical estimates with a common choice of $w_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(s)} = Q_{b,t-1}^{(s)}/Q_{b,t-1}$. The model confirms that the correlation level cannot predict Bitcoin returns while its change can. Therefore, these two tables quantitatively support our explanation on the Bitcoin return predictability puzzle.

Finally, Figure 6 visualizes how changes in non-speculative investors' correlation estimates and also demand can affect Bitcoin returns the next day. Inspired by the result of Table 13, we set $w_{b,t-1}^{(s)} = 0.6$ and $Q_{b,t-1}^{(s)} = Q_{b,t-1}^{(s)}/Q_{b,t-1} = 0.3$ and inherit other parameter values from the table. The graph is vertically shifted so that zero return means no change in correlation. Consistent with our intuition, Bitcoin returns decrease with correlation changes but increase with non-speculative investors' demand changes.

6.5 Discussion: What is this model about?

The model in this section and the proxy construction in Section 3 are simple but need clarification. First, our proxy construction is based on investors' sequential learning of conditional correlation estimates. However, the model in this section captures a snapshot of a static equilibrium while not specifying dynamic aspects of learning. Second, both the model and proxy are built on the mean-variance framework but fundamentally differ from CAPM. Investors are heterogeneous: they differ in what they learn, how their beliefs evolve, and the timing of their actions (asynchronous rebalancing). Therefore, the mechanism in our proxy construction and equilibrium model is unrelated to the canonical risk-premium-based explanation. We make additional clarification in the remainder of this section.

Is daily rebalancing a necessary assumption? No. First, we argue that our model can be interpreted in many different ways. The investors in the model equilibrium can be only a subset of investors. For example, non-speculative investors can be divided into h different subgroups. Then the investors in each subgroup rebalance their stock-and-Bitcoin portfolios every h trading days so that only one subgroup can rebalance on a given day. Our model works exactly in the same way even if we change our assumption on non-speculative investors this way, or even if we let non-

speculative investors have heterogeneous or even random rebalancing frequencies without altering the model outcomes. Likewise, speculative investors can periodically or randomly participate in the market. These alternative assumptions only result in a different interpretation on the model: the model is about the investors who participate in trading on a given day rather than all investors. Therefore, the quantitative implications remain the same in the tables and figures except for the definition or interpretation of parameters to calibrate, such as $w_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(s)}$. Furthermore, this alternative assumption of less frequent rebalancing explains why non-speculative investors do not mind one-day lag between rebalancing decisions and following order executions.

Can investors trade during a day, not only at closing? Yes, our model implications remain unaltered even if we allow investors in the model to trade during a day. Non-speculative investors (or their traders) aim to match the dollar amount of Bitcoin in the portfolios to $w_{b,t-1}^{(n)}A_{t-1}^{(n)}$, which is predetermined at $t - 1$ independently of the Bitcoin closing price $P_{b,t}$ at t , when they complete the last order at closing of day t . That is, non-speculative investors are free to trade during a day. Likewise, speculative investors also can freely trade during a day. Since the non-speculative investors will complete their final order at closing, variables such as $P_{b,t}$, μ_t^* , $w_{b,t}^{(s)}$, and $Q_{b,t}^{(s)}$ are also endogenously determined in the equilibrium at closing on day t . Therefore, whether investors can trade during a day does not affect our model outcomes.

7 Conclusion

The image of Bitcoin has transformed many times since its inception.⁴⁸ The recent COVID-19 episode only adds to it and we expect more in the future. Also, Bitcoin opens a gate for new class of assets, e.g., NFT (non-fungible token). Therefore, it is important to understand how a new narrative affects investors' behaviors and market equilibriums (Shiller 2020).

We present two empirical findings regarding this issue: (1) the time-varying ex-ante diversification benefit of Bitcoin as a recently promoted narrative and (2) the strong and robust Bitcoin return predictability that contradicts conventional wisdom. Then we explain the puzzling predictability by intuitive reasoning, empirical proxies, and an equilibrium model in light of trading practices and investors' learning. Our explanation emphasizes rational aspects of Bitcoin accompanied by

⁴⁸See [Biais et al. 2019](#) for an example of technical transformation.

institutional investors who diversify, as opposed to the most literature focusing on irrational retail traders' behaviors or price manipulation by Bitcoin whales.

Bitcoin return predictability in this paper differs from the return-predictability of stock markets. First, Bitcoin return predictability is unrelated to time-varying risk premium. Instead, investors' learning plays a crucial role in predictability, in particular, when the unknown is highly time-varying and so its uncertainty is high. Second, investors' incentive for learning and the resulting predictability are supposed to fade away eventually. The unknown correlation can become stable again; therefore, its uncertainty and learning efforts can dissipate. Even if the correlation remains time-varying, Bitcoin's next intended image-transformation may force investors to face a new type of uncertainty, and their attention moves from the correlation to something else.⁴⁹ However, the main idea of the paper is not about correlation *per se* but about general asset transformation by a new narrative (Yae et al. 2021) and investors' learning on it. Until Bitcoin becomes completely demystified, investors still learn about some characteristics of Bitcoin to reduce such a new kind of uncertainty. This new learning incentive and behavior, whatever they are, can offer another research opportunity.

We admit that our analysis has some limitations. First, the stylized model in Section 6 is purely static; it does not internalize investors' learning dynamics in equilibrium. Building a full dynamic model is a challenge because the model should also internalize all other non-conventional aspects of Bitcoin pricing, such as gradual adoption, mining cost, a fork, halving, and sentiments. Such a unified model would be too complicated to analyze and test, however. Second, we do not explain why correlations start to fluctuate as soon as Bitcoin futures trading begins. We only use such a structural break to split the sample for a main analysis (with post-futures sample) and a placebo test (with pre-futures sample). Third, our empirical proxies and theoretical model are based on the assumption of mean-variance investors. However, Bitcoin is well known for its high skewness and kurtosis (Conlon and McGee 2020). The first two moments have certainly a major effect, yet considering higher moments might lead to other interesting findings. Finally, our static model of Bitcoin is parsimonious but limited in applications. Yet, our speculative-vs-non-speculative framework can inspire more sophisticated Bitcoin pricing models, which we leave for future research.

⁴⁹For example, check out Bitcoin Vault at <https://yhoo.it/2Xh2Uoe>.

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Figure 1: Time-Varying Correlations Between Bitcoin and S&P500 Index

The figure shows the time-varying correlation estimates from DCC(1,1)-GARCH(1,1) models with daily log returns of Bitcoin and S&P500 Index: the gray regions (estimates from the full-sample) and the dashed lines (from two subsamples).

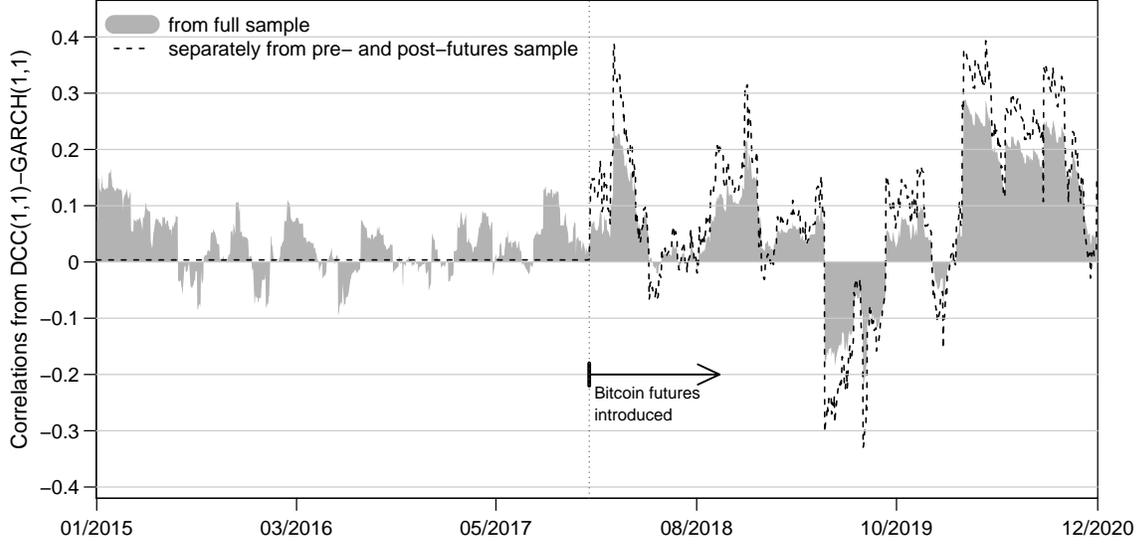


Figure 2: Bitcoin Demand due to Time-Varying Correlation and Volatility Ratio

The figure displays estimated excess non-speculative demands for Bitcoin for the post-futures sample: solid lines with round markers ($w_{b,t}^{(cor)} - \bar{w}_b$) and dashed line ($w_{b,t}^{(vol)} - \bar{w}_b$). The gray regions show Bitcoin's daily closing prices (USD).

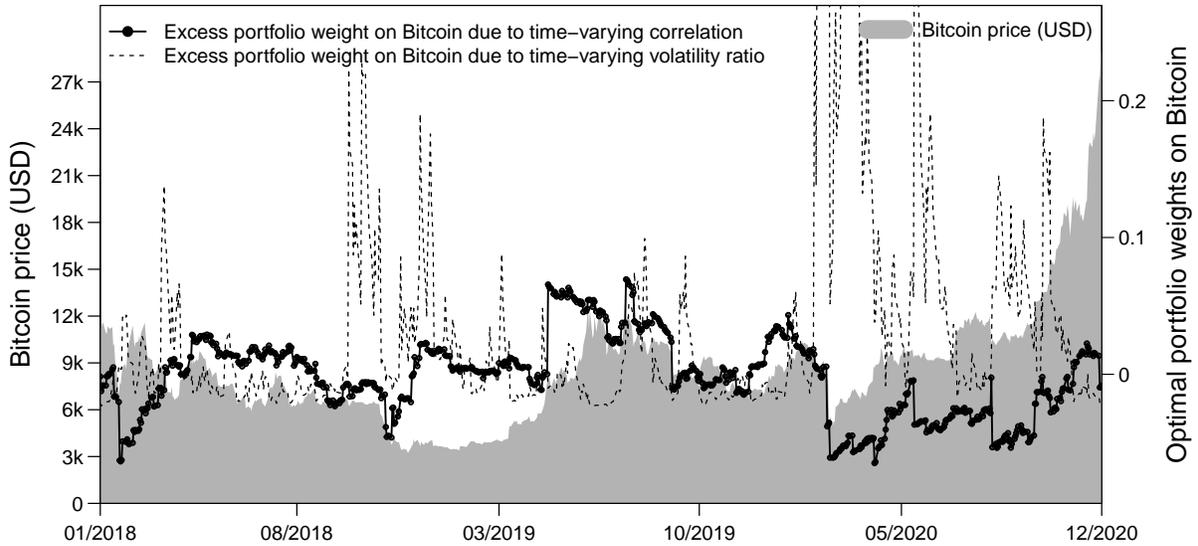


Figure 3: Variable Importance Heat Map

Two heat maps show variable importance metrics, which are reductions in R^2 when a given predictor is set to zero in a trained model, following Gu et al. (2020). The graphs show the relative importance of each predictor in each of the following machine learning algorithms: LASSO, Group LASSO, Elastic Net, Random Forest, Gradient Boosting Machine, and Neural Network with one hidden layer. In Panel (a), the variable importance metric (Metric I) is scaled by the column-wise sum so that the total sum for each column can be unity. Darker colors represent greater importance in prediction. Panel (b) repeats Panel (a) but scales the importance metric (Metric II) by normalizing the variable importance metric to be between zero and one for each algorithm. For Group LASSO, we use the groupings specified in Table 5 column ‘LG’ such that predictors with similar characteristics are grouped together.

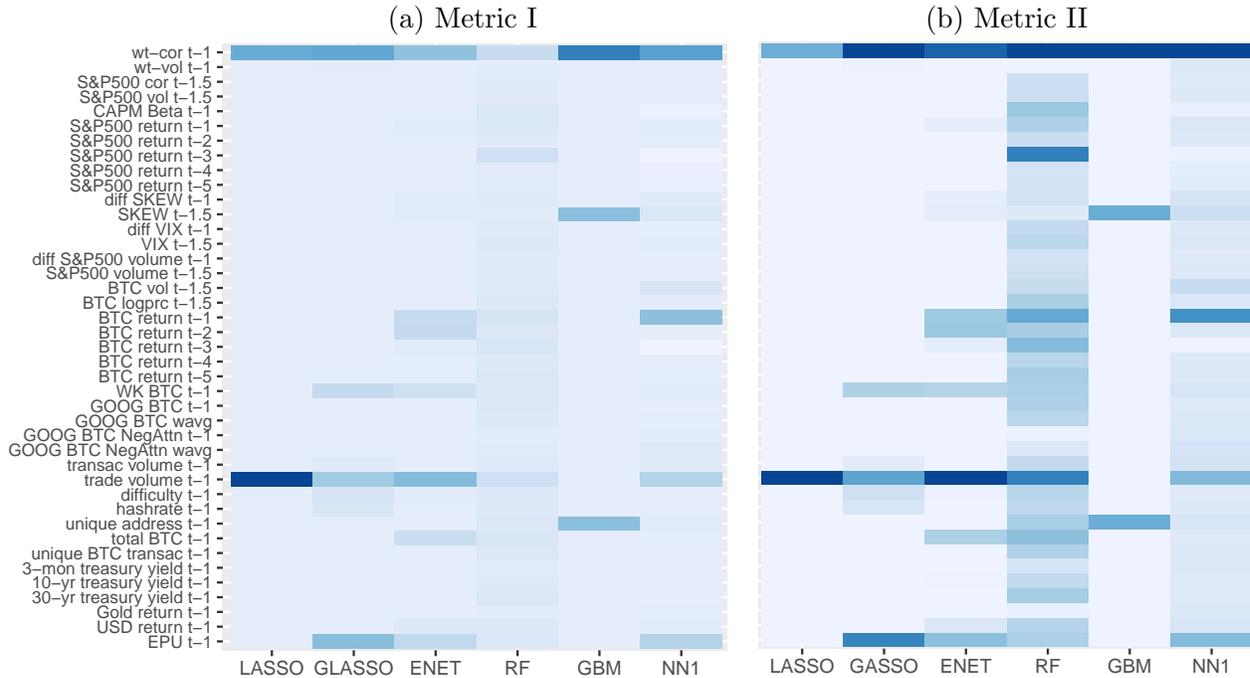


Figure 4: Return Predictability Conditional on Volatility Ratio

Panel (a) shows the sample standard deviation of $\Delta w_{b,(t-1):t}^{(cor)}$ for each choice of $\bar{\sigma}^*$ in the horizontal axis, expressed as its empirical quantile in the set $\{\sigma_t^*\}_{t=1}^T$. Panel (b) reports the Newey-West t-statistics of three different predictors $\Delta \rho_t$, $\Delta w_{b,(t-1):t}^{(cor)}$, and $\Delta w_{b,(t-1):t}^{(cor|vol)}$ from univariate predictive regressions with different trimming thresholds. The ratio of risk premium, $\bar{\mu}^* = RP(r_{b,t})/RP(r_{m,t})$, is fixed at unity.

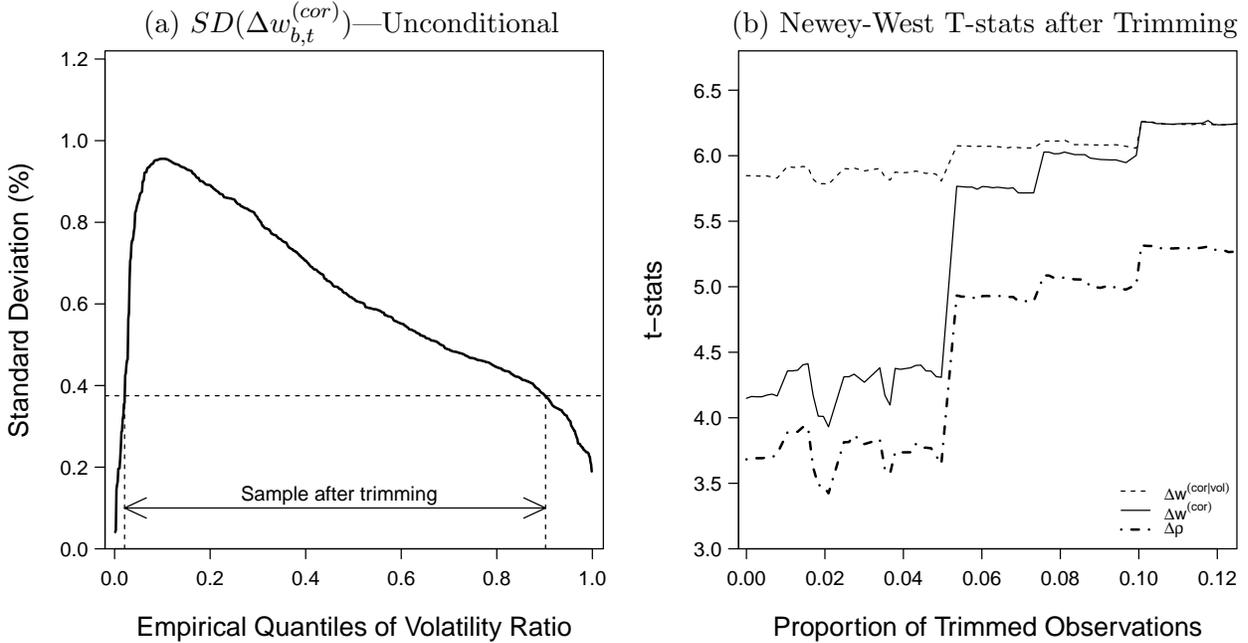


Figure 5: Return Predictability Conditional on Risk Premium Ratio

Two panels show how return predictability varies depending on an input value of $\vec{\mu}^*$ for optimal portfolio weight calculation, in terms of Newey West t-statistics in Panel (a) and R^2 in Panel (b) from univariate predictive regressions. The predictor variables are $\Delta w_{b,(t-1):t}^{(cor)}$, $\Delta w_{b,(t-1):t}^{(cor|vol)}$, $\Delta w_{b,(t-1):t}^{(vol)}$, and $\Delta w_{b,(t-1):t}^{(vol|cor)}$ where $\Delta w_{b,(t-1):t}^{(vol|cor)}$ is defined in a similar fashion to $\Delta w_{b,(t-1):t}^{(cor|vol)}$ in Equation (10).

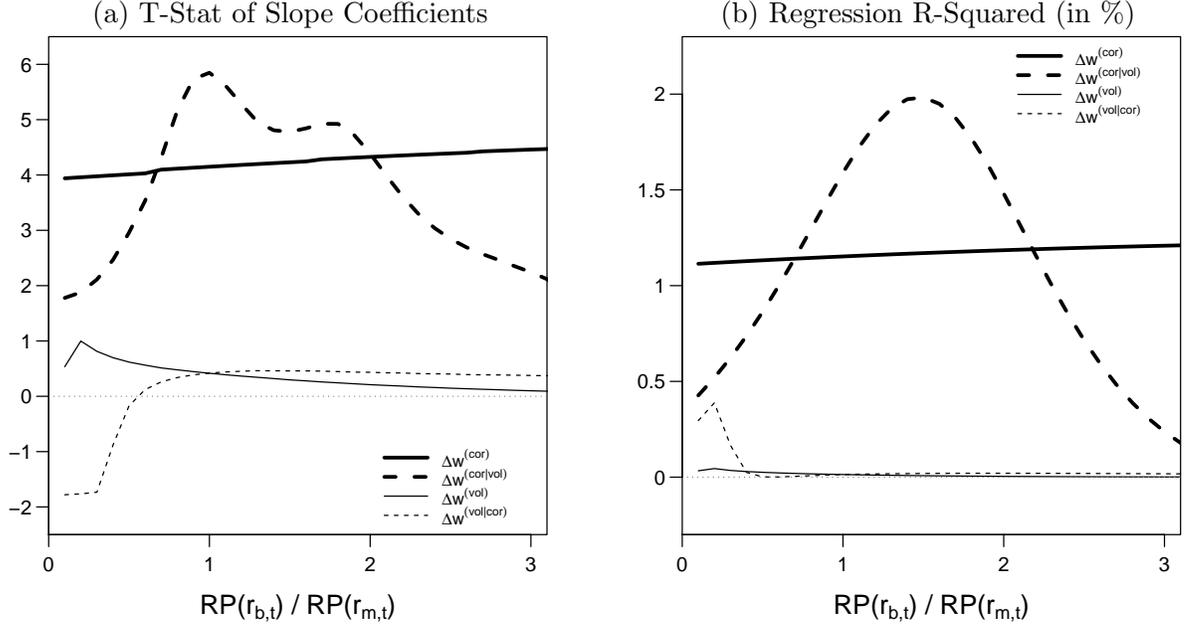


Figure 6: Comparative Statics Analysis of the Bitcoin Pricing Model

Panel (a) shows how Bitcoin price changes when the non-speculative investors' estimate on the correlation changes at static equilibrium as a comparative static analysis. The vertical axis is the subsequent log returns (%) of Bitcoin, in excess of log returns without correlation changes. Panel (b) repeats Panel (a) but replaces the horizontal axis by changes in non-speculative investors' optimal portfolio weights (%) on Bitcoin, corresponding to the correlation changes in Panel (a). We calibrate model parameters following Table 13, including $w_{b,t-1}^{(s)} = 0.6$ and $Q_{b,t-1}^{(s)}/Q_{b,t-1} = 0.3$.

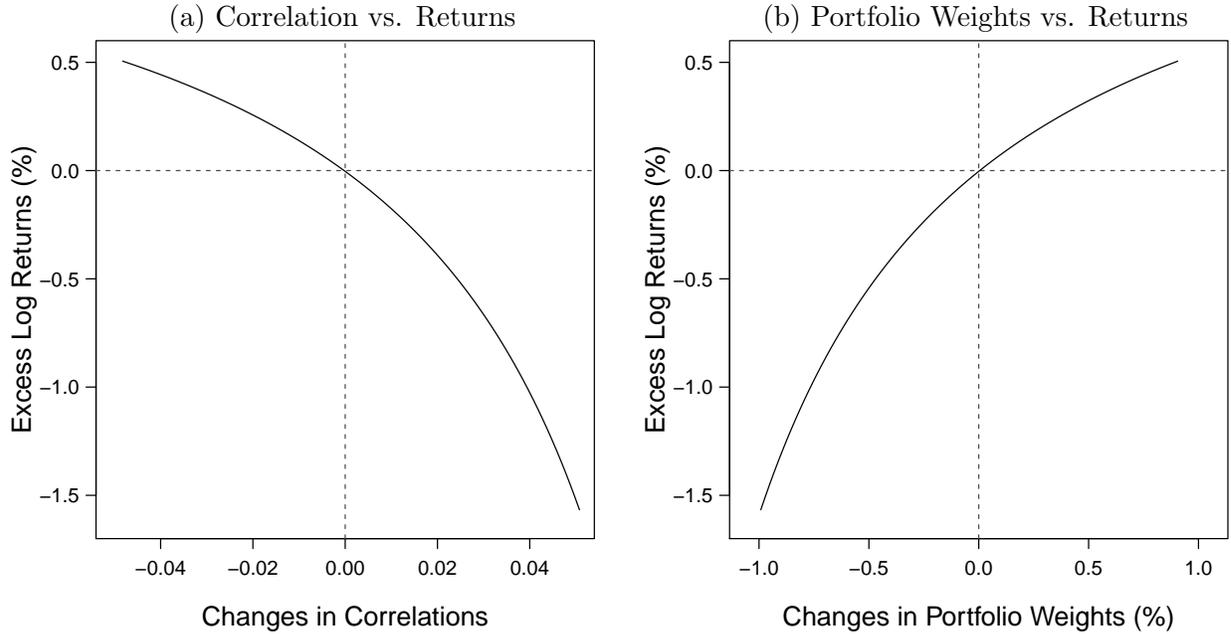


Table 1: Selected Parameter Estimates of DCC-GARCH Models

This table shows the estimates of parameters a and b and associated t-statistics in parenthesis by DCC(1,1)-GARCH(1,1) model for the pre-futures, the post-futures, and the full sample period. A full set of parameter estimates are reported in the online appendix [A.1](#).

Sample	Pre-futures	Post-futures	Full
from	01/01/2015	12/18/2017	01/01/2015
to	12/17/2017	12/31/2020	12/31/2020
<hr/>			
DCC(1,1)			
a	0.000 (0.00)	0.030 (2.01)	0.016 (1.52)
b	0.925 (2.94)	0.948 (30.84)	0.964 (28.84)

Table 2: Bitcoin Return Predictability

This table shows the results of univariate predictive regressions using data from post-futures period:

$$r_{b,t+1} = a_0 + a_1 x_t + \varepsilon_{t+1}$$

where $r_{b,t+1}$ is daily Bitcoin log returns and x_t can be any variable in the following. ρ_t is lagged Bitcoin-S&P500 correlation level estimated from DCC(1,1)-GARCH(1,1), β_t is lagged CAPM β , cov_t is lagged Bitcoin-S&P500 covariance, and $\Delta\rho_t$ is a lagged change in correlation. We compute β_t and cov_t using correlation and volatility estimates from DCC(1,1)-GARCH(1,1). All predictor variables x_t are not standardized. ‘pre-COVID’ indicates that regression is performed with data from 12/18/2017 to 02/29/2020. ‘LAD’ is short for least absolute deviations regression and ‘Rank’ refers to a rank-based estimation ([Hettmansperger and McKean 2010](#)). INT means both dependent variable and predictor adopt Inverse Normal Transformation (INT) based on [Beasley et al. \(2009\)](#). ‘Trimmed’ indicates that both dependent variable and predictor are trimmed at 2.5% and 97.5%. All t-statistics are Newey West t-statistics except for LAD and Rank where original t-statistics are used.

Predictor	coefficient	t-statistic	R^2
x_t	a_1	(NW)	(%)
ρ_t	0.001	0.049	0.000
β_t	-0.002	-0.735	0.063
cov_t	2.682	0.664	0.063
$\Delta\rho_t$	-0.153	-3.682	1.055
$\Delta\rho_t$ (pre-COVID)	-0.153	-2.888	1.079
$\Delta\rho_t$ (LAD)	-0.129	-3.816	1.017
$\Delta\rho_t$ (Rank)	-0.142	-4.085	1.039
$\Delta\rho_t$ (INT)	-0.134	-3.598	1.787
$\Delta\rho_t$ (Trimmed)	-0.245	-2.824	1.255

Table 3: Ex-ante Diversification Benefits of Bitcoin

This table shows the ex-ante diversification benefits of Bitcoin when it is mixed with S&P500 using correlation and volatility estimates from the DCC-GARCH model for different levels of $\bar{\mu}^*$. Sharpe ratio improvement is defined as $[SR_t(r_{p,t+1})/SR_t(r_{m,t+1}) - 1] \times 100$, where $r_{p,t}$ is a Bitcoin-stock portfolio return.

Among 765 days in the post-futures sample	Risk premium ratio, $\bar{\mu}^*$				
	1.0	1.5	2.0	2.5	3.0
Median of Sharpe raio improvement (%)	0.8	3.0	5.8	9.8	14.5
Proportion of positive portfolio weights on Bitcoin (%)	83.5	95.4	99.5	99.9	100.0

Table 4: Time-Dependency in Returns and Granger Causality Tests

Panel A shows OLS (ordinary least squares) estimates on the slope coefficient $a_1^{(L)}$ and Newey-West t-statistics of the following univariate predictive regression for the post-futures sample:

$$r_{b,t+1} = a_0 + a_1^{(L)} z_{t+1-L} + \varepsilon_{t+1},$$

where $r_{b,t+1}$ is daily Bitcoin log returns (%) and z_{t+1-L} is a lagged independent variable: Bitcoin demand changes due to correlation $\Delta w_{b,(t-L):t}^{(cor)}$ (standardized), Bitcoin returns $r_{b,t}$, or S&P500 returns $r_{m,t}$ for $L = 1, \dots, 4$. Each column in Panel A is from an individual univariate regression. On the other hand, Panel B shows three Granger causality test results in multivariate time-series setting:

$$r_{b,t+1} = a_0 + \sum_{L=1}^4 a_1^{(L)} z_{t+1-L} + \varepsilon_{t+1},$$

WLS refers to weighted least squares with weights that equal the inverses of time-varying variance estimates from DCC(1,1)-GARCH(1,1) fitting.

L (lag)	$\Delta w_{(t-L):(t-L+1)}^{(cor)}$				Bitcoin returns				S&P500 returns			
	1	2	3	4	1	2	3	4	1	2	3	4
Panel A: Univariate time-series regression												
$a_1^{(L)}$	0.51	0.21	-0.13	0.01	-0.07	0.08	0.05	-0.01	-0.15	0.10	0.02	-0.02
t-stat	4.15	0.48	-0.66	0.06	-1.49	1.87	1.31	-0.20	-1.00	1.07	0.11	-0.20
Panel B: Granger causality tests												
OLS coef.	0.43	0.21	-0.14	-0.07	-0.08	0.06	0.05	0.00				
t-stat	2.68	0.47	-0.82	-0.36	-1.76	1.16	1.25	0.07				
OLS coef.	0.42	0.21	-0.13	-0.08	-0.07	0.05	0.05	0.00	-0.09	0.03	0.00	-0.01
t-stat	2.64	0.49	-0.75	-0.42	-1.59	1.07	1.39	0.02	-0.67	0.30	0.01	-0.08
WLS coef.	0.65	0.25	-0.07	-0.06	0.01	0.04	0.07	0.02	0.13	0.06	-0.02	-0.07
t-stat	3.23	1.26	-0.33	-0.32	0.21	0.76	1.53	0.55	0.99	0.42	-0.19	-0.57

Table 5: A List of Predictors

The table shows a list of predictors used in Figure 3 and Table 6 and 7. When including both lagged change and level of the same variable, we compute the lagged level of the variable as the average of its one-day- and two-day-lagged values. Column ‘CG’ categorizes each variable into one of the followings: Group M (market attributes), Group B (blockchain attributes), and Group L (extra lagged returns). Column ‘LG’ specifies groupings of predictors for Group LASSO with those sharing an integer index belonging to a same group.

CG	LG	Predictor Name	Definition
	1	wt-cor t-1	change in optimal weight on Bitcoin from t-2 to t-1 due to time-varying correlation
	2	wt-vol t-1	change in optimal weight on Bitcoin from t-2 to t-1 due to time-varying volatility
	3	BTC return t-1	log return of Bitcoin at time t-1
	4	CAPM Beta t-1	CAPM Beta at time t-1
	5	trade volume t-1	Bitcoin trade volume on major exchanges at time t-1
	6	EPU t-1	Economic Policy Uncertainty Index at time t
M	1	S&P500 cor t-1.5	average of BTC-S&P500 correlation at time t-1 and t-2
M	2	S&P500 vol t-1.5	average of S&P500 volatility at time t-1 and t-2
M	7	S&P500 return t-1	log return of S&P500 index at time t-1
M	8	diff SKEW t-1	change of COBE SKEW index from t-2 to t-1
M	8	SKEW t-1.5	average of COBE SKEW index at time t-1 and t-2
M	9	diff VIX t-1	the difference between COBE VIX index at t-1 and that at t-2
M	9	VIX t-1.5	average of COBE VIX index at time t-1 and t-2
M	10	diff S&P500 volume t-1	change of CBOE Volume for S&P 500 Index options (as % of market cap) from t-2 to t-1
M	10	S&P500 volume t-1.5	average of CBOE Volume for S&P 500 Index options at time t-1 and t-2
M	2	BTC vol t-1.5	average of Bitcoin volatility at time t-1 and t-2
M	3	BTC logprc t-1.5	average of Bitcoin log price at time t-1 and t-2
M	11	3-mon treasury yield t-1	3-month treasury yield at time t-1
M	11	10-yr treasury yield t-1	10-year treasury yield at time t-1
M	11	30-yr treasury yield t-1	30-year treasury yield at time t-1
M	12	Gold return t-1	log return of Gold at time t-1
M	13	USD return t-1	log return of US Dollar at time t-1
B	14	WK BTC t-1	Wikipedia Bitcoin search counts at time t-1
B	14	GOOG BTC t-1	Google search counts for <i>Bitcoin</i> at time t-1
B	14	GOOG BTC wavg	average Google search counts for <i>Bitcoin</i> over the past week
B	14	GOOG BTC NegAttn t-1	ratio of Google search counts for <i>Bitcoin hack</i> to <i>Bitcoin</i> at t-1
B	14	GOOG BTC NegAttn wavg	average ratio of Google search counts for <i>Bitcoin hack</i> to <i>Bitcoin</i> over the past week
B	15	transac volume t-1	Bitcoin transaction volume at time t-1
B	16	difficulty t-1	Bitcoin mining difficulty at time t-1
B	16	hashrate t-1	Bitcoin hashrate at time t-1
B	17	unique address t-1	number of Bitcoin unique address at time t-1
B	18	total BTC t-1	total number of Bitcoin at time t-1
B	15	unique BTC transac t-1	number of unique Bitcoin transaction at time t-1
L	7	S&P500 return t-2	Daily log return of S&P500 index from t-2 to t-1
L	7	S&P500 return t-3	Daily log return of S&P500 index from t-3 to t-2
L	7	S&P500 return t-4	Daily log return of S&P500 index from t-4 to t-3
L	7	S&P500 return t-5	Daily log return of S&P500 index from t-5 to t-4
L	3	BTC return t-2	Daily log return of Bitcoin from t-2 to t-1
L	3	BTC return t-3	Daily log return of Bitcoin from t-3 to t-2
L	3	BTC return t-4	Daily log return of Bitcoin from t-4 to t-3
L	3	BTC return t-5	Daily log return of Bitcoin from t-5 to t-4

Table 6: In-Sample Predictive Regressions for Post-Futures Sample

The table shows the coefficient estimates and associated robust t-statistics in the predictive regression (7):

$$r_{b,t+1} = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + b_2 \Delta w_{b,(t-1):t}^{(vol)} + Z_t \gamma + \varepsilon_{t+1},$$

where $r_{b,t+1}$ is daily Bitcoin log returns (%). $\Delta w_{b,(t-1):t}^{(cor)}$ and $\Delta w_{b,(t-1):t}^{(vol)}$ refer to (standardized) changes in non-speculative Bitcoin demand due to correlation and volatility ratio changes from $t-1$ to t , respectively. Robust t-statistics in parenthesis are calculated from [Newey and West \(1987\)](#) and [Newey and West \(1994\)](#). Other predictor variables Z_t include lagged Bitcoin returns $r_{b,t} = \log(P_{b,t}) - \log(P_{b,t-1})$, lagged CAPM beta β_t , lagged Bitcoin trade volume $Volume_{b,t}$, Economic Policy Uncertainty index (policyuncertainty.com) $EPU_{b,t}$, and additional control variable groups. In Table 5 column ‘CG’, we classify each additional control variable into one of the following groups: Group M (market attributes), Group B (blockchain attributes), and Group L (extra lagged returns). When including a control variable group in regression, we use the first two principal components from that group. Note that investor attention variables related to Google search trends for Bitcoin are based on [Liu and Tsyvinski \(2021\)](#). Group L includes $r_{b,t-L}$ for $L = 1, \dots, 4$ and $r_{m,t-L}$ for $L = 0, \dots, 4$, implying Granger causality test.

Predictor	Post-futures (12/18/2017 to 12/31/2020)					Post-futures before COVID-19 (12/18/2017 to 02/29/2020)				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
$\Delta w_{b,(t-1):t}^{(cor)}$	0.51 (4.15)		0.47 (2.89)	0.47 (2.91)	0.47 (2.80)	0.50 (3.16)		0.48 (2.78)	0.47 (2.63)	0.48 (2.37)
$\Delta w_{b,(t-1):t}^{(vol)}$		0.06 (0.42)	0.03 (0.28)	0.04 (0.30)	0.03 (0.20)		-0.19 (-1.27)	-0.23 (-1.66)	-0.22 (-1.67)	-0.20 (-1.56)
$r_{b,t}$			-0.42 (-1.79)	-0.42 (-1.76)	-0.42 (-1.74)			-0.22 (-0.83)	-0.23 (-0.83)	-0.22 (-0.82)
β_t			0.04 (0.22)	0.06 (0.31)	0.07 (0.31)			0.06 (0.28)	0.07 (0.32)	0.09 (0.40)
$Volume_{b,t}$			-0.58 (-2.59)	-0.55 (-1.35)	-0.55 (-1.39)			-0.74 (-3.52)	-0.91 (-1.93)	-0.87 (-1.78)
$EPU_{b,t}$			0.63 (1.71)	0.64 (1.68)	0.63 (1.76)			0.21 (0.98)	0.20 (0.93)	0.21 (0.96)
Controls			M	MB	MBL			M	MB	MBL
R^2 (%)	1.15	0.01	3.94	3.99	4.00	1.17	0.16	4.04	4.14	4.38
$Adj.R^2$	1.02	-0.12	2.92	2.71	2.47	0.99	-0.02	2.62	2.36	2.24

Table 7: Importance of Predictor Variables

The table shows variable importance metrics (Metric I in Figure 3), which are reductions in R^2 when a given predictor is set to zero in a trained model (Gu et al. 2020), for $\Delta w_{b,(t-1):t}^{(cor)}$ and the second important predictor under different Machine Learning algorithms: LASSO, Group LASSO (GLASSO), Elastic Net (ENET), Random Forest (RF), Gradient Boosting Machine (GBM), and Neural Network with one hidden layer (NN1). The variable importance metric is scaled by the sum of all predictors so that the total sum for each algorithm can be unity. ‘MIV’ represents the most important variable whereas ‘SIV’ means the second most important variable. ‘VI score’ is a variable importance score implied by the variable importance metric.

	LASSO	GLASSO	ENET	RF	GBM	NN1
MIV	lagged trading volume	$\Delta w_{b,(t-1):t}^{(cor)}$	lagged trading volume	$\Delta w_{b,(t-1):t}^{(cor)}$	$\Delta w_{b,(t-1):t}^{(cor)}$	$\Delta w_{b,(t-1):t}^{(cor)}$
VI score of MIV	0.67	0.35	0.27	0.08	0.50	0.38
SIV	$\Delta w_{b,(t-1):t}^{(cor)}$	lagged EPU	$\Delta w_{b,(t-1):t}^{(cor)}$	lagged trading volume	lagged market skewness	lagged Bitcoin returns
VI score of SIV	0.33	0.26	0.24	0.06	0.25	0.24

Table 8: Predictive Regressions with Global Equities and Other Cryptos

This table repeats column (1) of Table 6 for alternative US stock market indexes, international equity markets, and three major cryptocurrencies: Bitcoin (BTC), Ethereum (ETH), and Ripple (XRP). We report coefficient estimates b_1 , their Newey-West statistics, and R^2 (%) of the predictive regressions with the post-futures sample from 12/18/2017 to 12/31/2020:

$$r_{c,t+1} = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + \varepsilon_{t+1} \quad \text{for } c \in \{BTC, ETH, XRP\},$$

where $r_{c,t+1}$ is daily log returns (%) and $\Delta w_{b,(t-1):t}^{(cor)}$ is (standardized) changes in non-speculative Bitcoin demand due to correlation from $t - 1$ to t . In DCC(1,1)-GARCH(1,1) estimation for each row, we use the following data: 1) S&P500, NYSE composite, and NASDAQ composite indexes for the US stock markets and 2) Shanghai Composite, Nikkei 225, BSE Sensex, FTSE 100, DAX, CAC 40, and FTSE Italia All-Share indexes for international equity markets. Once estimating $\Delta w_{b,(t-1):t}^{(cor)}$ individually from each international equity market, we aggregate such demands over different countries in the following ways. ‘Equal-weighted’ calculates a simple average of $\Delta w_{b,(t-1):t}^{(cor)}$ from S&P500 and the aforementioned seven international equity market indexes. ‘GDP-weighted’ and ‘Volume-weighted’ compute the averages, weighted by GDP (as of 2018) and cumulative Bitcoin trading volumes by country (as of July 2018 from <https://coin.dance>), respectively. Finally, ‘Combination forecast’ unites (with equal weights) the average predicted values of the $r_{c,t+1}$ from each individual predictive regression.

	Coefficient estimates			Newey-West t-stat			R^2 (%)		
	BTC	ETH	XRP	BTC	ETH	XRP	BTC	ETH	XRP
S&P500	0.51	0.81	0.68	4.15	4.10	2.75	1.15	1.75	0.83
NYSE	0.48	0.75	0.57	3.45	3.73	1.91	1.02	1.51	0.57
NASDAQ	0.50	0.86	0.61	4.20	4.32	2.61	1.07	1.99	0.66
China	0.55	0.80	0.78	3.29	3.23	2.97	1.30	1.64	1.08
Japan	0.59	0.68	0.86	2.83	2.37	2.80	1.42	1.13	1.22
India	0.96	1.15	0.86	2.05	1.83	1.97	4.00	3.47	1.32
UK	0.46	0.62	0.56	3.20	3.16	3.29	0.93	1.04	0.56
Germany	0.45	0.67	0.74	2.37	3.26	3.03	0.92	1.23	1.04
France	0.40	0.60	0.65	2.57	2.79	2.84	0.73	0.95	0.77
Italy	0.21	0.32	0.43	1.21	1.28	1.89	0.19	0.29	0.36
Equal-weighted	0.53	0.79	0.75	3.55	3.96	3.25	1.25	1.65	1.01
GDP-weighted	0.62	0.94	0.86	4.58	4.74	3.68	1.71	2.38	1.34
Volume-weighted	0.61	0.92	0.80	4.91	4.85	3.50	1.62	2.24	1.16
Combination forecast	0.60	0.86	0.85	3.97	4.39	3.64	1.56	1.98	1.28

Table 9: Out-Of-Sample Predictive Regressions

This table repeats column (1) of Table 6 in a completely out-of-sample fashion. The reported numbers are in- and out-of-sample performance measures of the predictive regression:

$$r_{b,t+1} = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + \varepsilon_{t+1}.$$

At each day t , we re-estimate $\Delta x_{b,(t-1):t}$ (from DCC-GARCH) and (b_0, b_1) while expanding a training window without any look-ahead biases. We compare four different estimation methods: ordinary least squares (OLS), weighted least squares (WLS), least absolute deviation (LAD, median), and rank-based estimation (Rank) from [Hettmansperger and McKean \(2010\)](#). For WLS weights, we use reciprocals of real-time time-varying variance estimates from DCC(1,1)-GARCH(1,1). R_{IS}^2 is in-sample R^2 . R_{OS}^2 is defined in Equation (8) following [Campbell and Thompson \(2008\)](#) while R_{OS}^{2*} is defined in Equation (9) following [Gu et al. \(2020\)](#).

Evaluation period	1/2/2019 to 12/31/2020				1/2/2019 to 2/29/2020			
	OLS	WLS	LAD	Rank	OLS	WLS	LAD	Rank
$\Delta w_{b,(t-1):t}^{(cor)}$, from S&P500								
R_{IS}^2 (%)	1.29	1.17	0.92	0.97	1.47	1.36	1.07	1.07
R_{OS}^2 (%)	1.07	1.54	2.32	2.33	1.49	2.07	3.49	3.52
R_{OS}^{2*} (%)	-0.08	0.40	1.19	1.20	-0.55	0.04	1.49	1.52

Table 10: Expected Errors in Portfolio Weights

The table shows the following AR(1) model estimation:

$$z_t = b_0 + b_1 z_{t-1} + e_t,$$

where z_t is either DCC(1,1)-GARCH(1,1) estimate (σ_t^* or ρ_t) or the portfolio weight ($w_{b,t}^{(vol)}$ or $w_{b,t}^{(cor)}$) due to learning, for the post-futures sample. Standard errors are reported in parenthesis. In the last row, we report $var(e_t^{(\sigma)})/var(e_t^{(\rho)})$ as a measure of uncertainty ratio. For comparison, σ_t^* and ρ_t are standardized.

z_t	DCC-GARCH est.		Portfolio weights	
	σ_t^*	ρ_t	$w_{b,t}^{(vol)}$	$w_{b,t}^{(cor)}$
b_1	0.92 (0.01)	0.98 (0.01)	0.92 (0.01)	0.98 (0.01)
$var(e_t)$	0.16	0.05	11.13	0.37
$\frac{var(e_t^{(\sigma)})}{var(e_t^{(\rho)})}$		3.48		30.12

Table 11: Comparison of Predictability by Different Proxies (In-Sample)

The table shows the coefficient estimates and associated robust t-statistics in univariate predictive regressions by OLS (ordinary least squares):

$$r_{b,t+1} = b_0 + b_1 \Delta x_t + \varepsilon_{t+1},$$

where $r_{b,t+1}$ is daily Bitcoin log returns (%) and x_t can be $\Delta \rho_t$, $\Delta w_{b,(t-1):t}^{(cor)}$, or $\Delta w_{b,(t-1):t}^{(cor|vol)}$. All predictor variables are standardized by their sample means and standard deviations, respectively. Robust t-statistics in parenthesis are calculated from [Newey and West \(1987\)](#) and [Newey and West \(1994\)](#). The estimations are based on the post-futures sample (12/18/2017 to 12/31/2020). The last column shows the correlation coefficients of a proxy to $\Delta \rho_t$.

Predictor	(1)	(2)	(3)	(4)	(5)	(6)	(7)	Corr. to $\Delta \rho_t$
$\Delta \rho_t$	-0.49 (-3.68)			1.11 (1.04)	0.51 (1.04)		0.10 (0.09)	1.00
$\Delta w_{b,(t-1):t}^{(cor)}$		0.51 (4.15)		1.62 (1.55)		-0.63 (-0.98)	-0.52 (-0.36)	-0.99
$\Delta w_{b,(t-1):t}^{(cor vol)}$			0.60 (5.85)		1.08 (2.26)	1.20 (1.93)	1.19 (1.78)	-0.93
R^2 (%)	1.05	1.15	1.59	1.24	1.75	1.76	1.76	
$Adj.R^2$ (%)	0.92	1.02	1.46	0.98	1.49	1.50	1.37	

Table 12: Comparison of Predictability by Different Proxies (Out-Of-Sample)

This table repeats out-of-sample prediction in Table 9 for three predictors, $\Delta\rho_{(t-1):t}$, $\Delta w_{b,(t-1):t}^{(cor)}$, and $\Delta w_{b,(t-1):t}^{(cor|vol)}$ in Table 11 using LAD and Rank regressions. The predictor variables are either estimated with S&P500 or aggregated from individual estimations with global equity indexes with GDP weights ('GDP-weighted' in Table 8). R_{OS}^2 and R_{OS}^{2*} are as defined in Equation (8) and (9), respectively. R_{IS}^2 is in-sample R^2 .

Predictors	LAD			Rank		
	$\Delta\rho_{(t-1):t}$	$\Delta w_{b,(t-1):t}^{(cor)}$	$\Delta w_{b,(t-1):t}^{(cor vol)}$	$\Delta\rho_{(t-1):t}$	$\Delta w_{b,(t-1):t}^{(cor)}$	$\Delta w_{b,(t-1):t}^{(cor vol)}$
from S&P500						
R_{IS}^2	0.78	0.92	1.15	0.78	0.97	1.29
R_{OS}^2	2.13	2.32	1.46	2.17	2.33	2.56
R_{OS}^{2*}	1.00	1.19	1.34	1.04	1.20	1.43
Globally aggregated (GDP)						
R_{IS}^2	1.01	1.26	1.71	0.99	1.16	1.81
R_{OS}^2	2.46	2.86	3.27	2.55	2.84	3.40
R_{OS}^{2*}	1.34	1.74	2.15	1.42	1.71	2.29

Table 13: Model-Implied Coefficients in Predictive Regressions (Corr. Changes)

This table shows the model-implied coefficient:

$$a_1^{(c)} = \frac{1}{P_{b,t}} E \left[\frac{\partial P_{b,t+1}}{\partial \rho_t} \right] = E \left[\frac{\partial \log P_{b,t+1}}{\partial \rho_t} \right],$$

which corresponds to the slope coefficient $a_1^{(c)}$ in the predictive regression of Bitcoin returns $r_{b,t+1}$ on $(\rho_t - \rho_{t-1})$, i.e., changes in correlation between Bitcoin and the stock market returns:

$$r_{b,t+1} = a_0^{(c)} + a_1^{(c)}(\rho_t - \rho_{t-1}) + \varepsilon_{t+1}$$

We compute $a_1^{(c)}$ using a numerical derivative for each combination of $w_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(n)}/Q_{b,t-1}$. At day $t-1$, speculative investors have the optimal weight on Bitcoin $w_{b,t-1}^{(s)}$ at $t-1$. Also, at $t-1$, speculative and non-speculative investors who participate in trading at t hold $Q_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(n)}$ units of Bitcoin, respectively, where $Q_{b,t-1} = Q_{b,t-1}^{(s)} + Q_{b,t-1}^{(n)}$. The point estimate on $a_1^{(c)}$ of the regression using our data is -0.153 where its 95% confidence interval implied by Newey-West standard errors is $(-0.236, -0.072)$. Therefore, we can reverse-engineer two parameters $w_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(s)}/Q_{b,t-1}$ by matching the point estimate to the numbers in the table. For example, calibrating $w_{b,t-1}^{(s)} = 0.6$ and $Q_{b,t-1}^{(s)}/Q_{b,t-1} = 0.3$ produces $a_1^{(c)} = -0.154$ if other parameters are calibrated at their empirically representative values as in Section 6.2.

$w_{b,t-1}^{(s)}$	$Q_{b,t-1}^{(s)}/Q_{b,t-1}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	-0.053	-0.024	-0.014	-0.009	-0.006	-0.004	-0.003	-0.001	-0.001
0.2	-0.117	-0.053	-0.031	-0.020	-0.013	-0.009	-0.006	-0.003	-0.001
0.3	-0.195	-0.088	-0.052	-0.033	-0.022	-0.015	-0.010	-0.006	-0.002
0.4	-0.291	-0.133	-0.078	-0.051	-0.034	-0.023	-0.015	-0.009	-0.004
0.5	-0.408	-0.189	-0.112	-0.072	-0.048	-0.032	-0.021	-0.012	-0.005
0.6	-0.554	-0.260	-0.154	-0.100	-0.067	-0.045	-0.029	-0.017	-0.008
0.7	-0.736	-0.350	-0.209	-0.136	-0.091	-0.061	-0.040	-0.023	-0.010
0.8	-0.964	-0.467	-0.281	-0.184	-0.124	-0.083	-0.054	-0.031	-0.014
0.9	-1.253	-0.621	-0.377	-0.248	-0.167	-0.113	-0.073	-0.043	-0.019

Table 14: Model-Implied Coefficients in Predictive Regressions (Corr. Levels)

This table repeats Table 13 but instead shows the model-implied coefficient:

$$a_1^{(l)} = \frac{E[r_{b,t+1}^{(+)}] - E[r_{b,t+1}^{(-)}]}{\rho_t^{(+)} - \rho_t^{(-)}},$$

where $\rho_t^{(+)} = \bar{\rho}^* + SD(\rho_t)$, $\rho_t^{(-)} = \bar{\rho}^* - SD(\rho_t)$, $(\bar{\rho}^*, \rho_t)$ are the unconditional and conditional correlations from the DCC-GARCH estimation, respectively, and $SD(\rho_t) \approx 0.15$. On the other hand, $E[r_{b,t+1}^{(+)}]$ and $E[r_{b,t+1}^{(-)}]$ are the model-implied Bitcoin log returns corresponding to the correlation level $\rho_t^{(+)}$ and $\rho_t^{(-)}$, respectively. The coefficient $a_1^{(l)}$ corresponds to the slope coefficient in the predictive regression of Bitcoin returns $r_{b,t+1}$ on ρ_t , i.e., levels of correlation between Bitcoin and the stock market returns:

$$r_{b,t+1} = a_0^{(l)} + a_1^{(l)} \rho_t + \varepsilon_{t+1}$$

To compute $E[r_{b,t+1}^{(+)}]$, we first find $E[\rho_{t+1} | \rho_t^{(+)}]$ from Equation (1). Then compute the equilibrium Bitcoin prices at $t + 1$ when $\rho_t = E[\rho_{t+1} | \rho_t^{(+)}]$ and $\rho_t = \rho_t^{(+)}$, respectively, holding others constant. Finally, we have a gross return that equals the ratio of those two prices, and its logarithm is $E[r_{b,t+1}^{(+)}]$. We compute $E[r_{b,t+1}^{(-)}]$ in a similar fashion. The point estimate on $a_1^{(l)}$ of the regression using our data is practically zero, 0.0006, where its 95% confidence interval implied by Newey-West standard errors is $(-0.0946, +0.0957)$. Therefore, the model can simultaneously generate both $a_1^{(c)}$ and $a_1^{(l)}$ matching their empirical estimates with a common choice of $w_{b,t-1}^{(s)}$ and $Q_{b,t-1}^{(s)} = Q_{b,t-1}^{(s)} / Q_{b,t-1}$.

$w_{b,t-1}^{(s)}$	$Q_{b,t-1}^{(s)} / Q_{b,t-1}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.0003	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.0006	0.0003	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.3	0.0011	0.0004	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
0.4	0.0017	0.0007	0.0004	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000
0.5	0.0027	0.0010	0.0006	0.0004	0.0002	0.0002	0.0001	0.0001	0.0000
0.6	0.0040	0.0015	0.0008	0.0005	0.0003	0.0002	0.0001	0.0001	0.0000
0.7	0.0059	0.0021	0.0011	0.0007	0.0004	0.0003	0.0002	0.0001	0.0000
0.8	0.0085	0.0030	0.0016	0.0010	0.0006	0.0004	0.0003	0.0001	0.0001
0.9	0.0120	0.0042	0.0022	0.0013	0.0008	0.0006	0.0003	0.0002	0.0001

Online Appendix

(Not Intended for Publication)

for

Sequential Learning, Asset Allocation, and Bitcoin Returns

A.1 DCC-GARCH Models

We also consider alternative GARCH models with conditional correlations. The Exponential Generalized Autoregressive Conditional Heteroscedastic (EGARCH) model (Nelson 1991) can capture the leverage effects in asset returns — asymmetric responses of conditional variances to negative or positive shocks. On the other hand, an asymmetric response to joint shocks in conditional correlations can be captured by the Asymmetric Generalized Dynamic Conditional Correlation (AG-DCC) model (Cappiello et al. 2006). As a special case, an Asymmetric DCC (A-DCC) model modifies the DCC equation as follows.

$$\mathbf{Q}_t = (\bar{\mathbf{R}} - a^2 \bar{\mathbf{R}} - b^2 \bar{\mathbf{R}} - g^2 \bar{\mathbf{N}}) + a^2 \boldsymbol{\epsilon}_{t-1} \boldsymbol{\epsilon}'_{t-1} + g^2 \mathbf{n}_{t-1} \mathbf{n}'_{t-1} + b^2 \mathbf{Q}_{t-1}$$

where a , b , and g are scalar parameters; $\bar{\mathbf{N}} = E[\mathbf{n}_t \mathbf{n}'_t]$ and $\mathbf{n}_t = I[\boldsymbol{\epsilon}_t < 0] \circ \boldsymbol{\epsilon}_t$ where $I[\cdot]$ is an indicator function and “ \circ ” represents Hadamard product.

Table A.1: Parameter Estimates of DCC-GARCH Models

The table shows parameter estimates and associated t-statistics in parenthesis for different DCC-GARCH models. E(1,1) refers to EGARCH(1,1) and a(1,1) refers to Asymmetric Dynamic Conditional Correlation a-DCC(1,1). Panel A, B, and C are for full, pre-futures, and post-futures sample periods, respectively. Panel A and B only report DCC parameters a and b while Panel C also includes GARCH parameter estimates and information criteria. DCC- a estimates are all zeros in the pre-futures period and become positive and overall significant in the post-futures period across all DCC-GARCH models. This result implies a constant correlation in the pre-futures period and time-varying correlations in the post-futures period.

DCC order	(1,1)	(1,1)	(1,1)	(1,1)	a(1,1)
GARCH order	(1,1)	E(1,1)	(2,1)	(1,2)	(1,1)
<i>Panel A: Full sample period, 01/01/2015–12/31/2020</i>					
<u>DCC</u>					
a	0.016 (1.52)	0.014 (1.61)	0.016 (1.43)	0.016 (1.43)	0.015 (1.94)
b	0.964 (28.84)	0.970 (37.78)	0.963 (26.55)	0.963 (26.24)	0.960 (27.13)
<i>Panel B: Pre-futures period, 01/01/2015–12/17/2017</i>					
<u>DCC</u>					
a	0.000 (0.00)	0.000 (0.00)	0.000 (0.00)	0.000 (0.00)	0.000 (0.00)
b	0.925 (2.94)	0.926 (2.85)	0.924 (3.77)	0.926 (3.51)	0.942 (5.53)
<i>Panel C: Post-futures period, 12/18/2017–12/31/2020</i>					
<u>DCC</u>					
a	0.030 (2.01)	0.025 (1.88)	0.034 (1.94)	0.032 (1.99)	0.028 (1.88)
b	0.948 (30.84)	0.957 (33.67)	0.941 (26.86)	0.944 (28.54)	0.948 (30.47)
g					0.007 (0.37)
<u>GARCH-Bitcoin</u>					
α_1	0.192 (1.96)	-0.127 (-1.17)	0.037 (0.76)	0.223 (1.80)	0.192 (1.96)
α_2			0.266 (1.66)		
β_1	0.703 (10.86)	0.826 (18.74)	0.522 (4.59)	0.1621 (0.96)	0.703 (10.85)
β_2				0.480 (3.52)	
γ_1		0.353 (1.98)			
<u>GARCH-S&P500</u>					
α_1	0.285 (3.55)	-0.152 (-3.06)	0.231 (1.76)	0.285 (2.14)	0.285 (3.56)
α_2			0.093 (1.09)		
β_1	0.714 (9.88)	0.936 (53.32)	0.675 (4.85)	0.715 (1.08)	0.714 (9.88)
β_2				0.000 (0.00)	
γ_1		0.450 (4.93)			
<u>Information Criteria</u>					
AIC	-9.78	-9.80	-9.81	-9.78	-9.78
BIC	-9.72	-9.72	-9.73	-9.70	-9.71

A.2 Demand Monotonicity

Intuitively, portfolio diversification implies that the demand for Bitcoin $w_{b,t}$ should monotonically decrease with the correlation between Bitcoin and the stock (ρ_t), and also with Bitcoin-stock volatility ratio σ_t^* . In particular, if $\mu_t^* = 1$ and $\rho_t = 0$, then $w_{b,t} = [\{\sigma_t^*\}^2 + 1]^{-1}$ decreases with σ_t^* . If $\mu_t^* = 1$ and $\sigma_t^* = 2$, then $w_{b,t} = \frac{1}{2} - \frac{3}{8} [\frac{5}{4} - \rho_t]^{-1}$ decreases with ρ_t . Here, we formalize conditions for demand monotonicity to elucidate our Bitcoin demand estimation procedure.

First, note the denominator in Equation (2) is positive if and only if

$$\rho_t < \frac{\mu_t^* + \{\sigma_t^*\}^2}{(\mu_t^* + 1)\sigma_t^*} \quad (\text{A.1})$$

The optimal weight $w_{b,t}$ diverges when its denominator is zero, but we find that reasonable values of inputs generate positive denominator, as shown later in Figure A.1 Panel (a).

Proposition A.1. (Demand Monotonicity) *Suppose that the condition (A.1) is satisfied. Then we have the following necessary and sufficient conditions.*

- (a) *The optimal portfolio weight $w_{b,t}$ on Bitcoin monotonically decreases with Bitcoin-Stock correlation coefficient ρ_t if and only if*

$$\frac{\mu_t^*}{\sigma_t^*} = \frac{\text{Sharpe Ratio}_t(\text{Bitcoin})}{\text{Sharpe Ratio}_t(\text{Market})} < 1 \quad (\text{A.2})$$

- (b) *The optimal portfolio weight $w_{b,t}$ on Bitcoin monotonically decreases with Bitcoin-to-Stock volatility ratio σ_t^* if and only if*

$$\rho_t < \frac{2\mu_t^*\sigma_t^*}{\{\mu_t^*\}^2 + \{\sigma_t^*\}^2} \quad (\text{A.3})$$

The condition (A.2) should be satisfied in the CAPM world where the market portfolio has the highest possible Sharpe ratio. Even if CAPM is not true, it is unlikely that highly volatile Bitcoin has higher Sharpe ratio than the well-diversified stock market.^{A.1} Also, the risk premium of Bitcoin should be higher than the stock market. Otherwise, no one will invest in Bitcoin with such a high

^{A.1}Note the historical realized Sharpe ratio is not the right measure in Proposition A.1. We need ex-ante Sharpe ratios which we cannot observe directly.

volatility. That is, we presume that the condition (A.2) and the following condition are satisfied.

$$1 \leq \mu_t^* = \frac{\mu_{b,t}}{\mu_{m,t}} \leq \frac{\sigma_{b,t}}{\sigma_{m,t}} \quad (\text{A.4})$$

The daily conditional risk premium or the risk premium ratio μ_t^* is difficult to estimate. Instead, we investigate later how μ_t^* , within the inequality condition (A.4), affect the demand monotonicity condition (in Figure A.1) and our main empirical results (in Figure A.7).

We notice that Bitcoin demand monotonicity conditions are well satisfied in the post-futures period. In Table A.3, the correlation coefficient between $w_{b,t}^{(cor)}$ and $Corr_t(r_b, r_m)$ is -0.997 , implying that they are truly monotonically associated, as expected by Proposition A.1. Also, Figure A.1 visualizes how well the condition (5) and the condition (A.3) in Proposition A.1 (b) are satisfied by the conditional correlation and volatility ratio estimates from the DCC(1,1)-GARCH(1,1) model fitting with the post-futures sample. Panel (a) confirms that all the correlation and volatility ratio estimates are well located within the gray region, implying the condition (A.1) is met. Therefore, the demand for Bitcoin monotonically decreases with the correlation (ρ_t) between Bitcoin and the stock market in the post-futures period. Similarly, Panel (b) shows that most of the correlation and volatility ratio estimates are located inside the gray region even in the most restricted case $\mu_t^* = 1$, that is, the case where the risk premia of Bitcoin and the stock market are identical. Hence, the demand for Bitcoin monotonically decreases also with Bitcoin-stock volatility ratio (σ_t^*). Nonetheless, our analysis does not require the demand monotonicity conditions, because we look into how Bitcoin prices react to different components in the Bitcoin demand proxy $w_{b,t}$ rather than the conditional correlation and volatilities.

A.3 Proof of Proposition A.1

The optimal portfolio weight on Bitcoin is

$$w_{b,t} = \frac{\mu_t^* - \rho_t \sigma_t^*}{(\mu_t^* - \rho_t \sigma_t^*) + (\sigma_t^* - \rho_t \mu_t^*) \sigma_t^*} = \frac{\mu_t^* - \rho_t \sigma_t^*}{g(\mu_t^*, \sigma_t^*, \rho_t)} \quad (\text{A.5})$$

1) Proof of Proposition A.1(a): Suppose both conditions (A.1) and (A.2) are satisfied. The partial derivative of $w_{b,t}$ with respect to ρ_t can be calculated as follows and will be negative.

$$\frac{\partial w_{b,t}}{\partial \rho_t} = \frac{-\sigma_t^* g(\mu_t^*, \sigma_t^*, \rho_t) + (\mu_t^* - \rho_t \sigma_t^*)(\sigma_t^* + \sigma_t^* \mu_t^*)}{[g(\mu_t^*, \sigma_t^*, \rho_t)]^2} = \frac{\{\mu_t^*\}^2 - \{\sigma_t^*\}^2}{[g(\mu_t^*, \sigma_t^*, \rho_t)]^2} \sigma_t^* < 0$$

The last inequality comes from $\sigma_t^* > 0$ and the condition (A.2) that implies $0 < \mu_t^* < \sigma_t^*$. Note we need the condition (A.1) to guarantee the denominator of the weight positive: $g(\mu_t^*, \sigma_t^*, \rho_t) > 0$. It is because $w_{b,t}$ diverges at $g(\mu_t^*, \sigma_t^*, \rho_t) = 0$. Therefore, $w_{b,t}$ monotonically decreases with ρ_t ■

2) Proof of Proposition A.1(b) Suppose both conditions (A.1) and (A.3) are satisfied. The partial derivative of $w_{b,t}$ with respect to σ_t^* can be calculated as follows and will be negative.

$$\frac{\partial w_{b,t}}{\partial \sigma_t^*} = \frac{-\rho_t g(\mu_t^*, \sigma_t^*, \rho_t) - (\mu_t^* - \rho_t \sigma_t^*)(-\rho_t + 2\sigma_t^* - \rho_t \mu_t^*)}{[g(\mu_t^*, \sigma_t^*, \rho_t)]^2} = \frac{\rho_t(\{\mu_t^*\}^2 + \{\sigma_t^*\}^2) - 2\mu_t^* \sigma_t^*}{[g(\mu_t^*, \sigma_t^*, \rho_t)]^2} < 0 \quad (\text{A.6})$$

The last inequality comes from the condition (A.3). Note we need the condition (A.1) to guarantee the denominator of the weight positive: $g(\mu_t^*, \sigma_t^*, \rho_t) > 0$. It is because $w_{b,t}$ diverges at $g(\mu_t^*, \sigma_t^*, \rho_t) = 0$. Therefore, $w_{b,t}$ monotonically decreases with σ_t^* ■

Figure A.1: Demand Monotonicity Conditions

Three curves in Panel (a) draw the right-hand-side of the inequality (A.1) at three different μ_t^* levels at 1, 2, and 3.26 which equals the estimate on σ_t^* from the data, consistent with the inequality (A.4). The areas below the curves are therefore where the condition (A.1) is met. Similarly, three curves in Panel (b) draw the right-hand-side of the inequality (A.3) at three different μ_t^* levels at 1, 2, and 3.22. In both panels, horizontal axis is the ratio of Bitcoin and S&P500's conditional volatilities while the vertical axis is their conditional correlations. The black dots in both panels represent corresponding observations in the post-futures period.

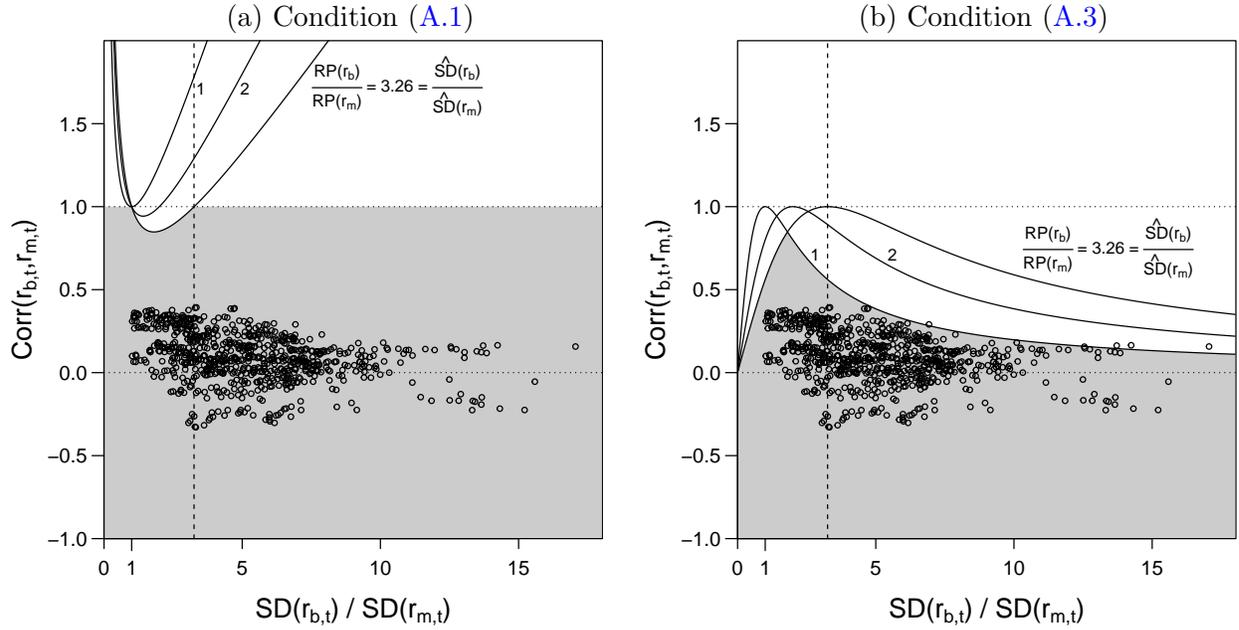


Table A.2: Summary Statistics of Data & Non-Speculative Bitcoin Demand

The table shows summary statistics and the non-speculative Bitcoin demand estimates implied by DCC(1,1)-GARCH(1,1) for the post-futures sample (12/18/2017-12/31/2020) and the post-futures sample before the COVID-19 period (12/18/2017-02/29/2020). All measures of returns, standard deviations, and weights are expressed in percentages whereas other estimates are unit-free. r_m and r_b denote daily log returns of S&P500 index and Bitcoin (USD), respectively. $w_{b,t}^{(cor)}$ is the non-speculative demand due to correlation changes while $w_{b,t}^{(vol)}$ is the non-speculative demand due to volatility ratio changes. $w_{b,t}^{(c+v)}$ is the non-speculative demand due to both correlation and volatility ratio changes. All variables are at daily frequency.

	Post-Futures					Post-Futures before COVID-19				
	Mean	S.D.	Skew	Min	Max	Mean	S.D.	Skew	Min	Max
<u>Asset returns</u>										
Bitcoin (r_b , %)	0.06	4.79	-1.3	-46.47	20.30	-0.13	4.65	-0.3	-23.87	20.30
S&P500 (r_m , %)	0.04	1.47	-1.0	-12.77	8.97	0.02	0.96	-0.8	-4.52	4.84
<u>DCC-GARCH est.</u>										
$Corr_t(r_b, r_m)$	0.10	0.15	-0.4	-0.33	0.39	0.04	0.13	-0.5	-0.33	0.39
$SD_t(r_b)$	4.68	1.67	3.6	3.18	21.00	4.68	1.41	1.7	3.19	11.89
$SD_t(r_m)$	1.17	0.98	4.0	0.44	9.33	0.93	0.45	1.7	0.44	3.13
$SD_t(r_b)/SD_t(r_m)$	5.25	2.68	1.0	1.02	17.06	5.90	2.63	1.1	1.33	17.06
CAPM β_t	0.37	0.76	-1.1	-3.43	2.68	0.24	0.83	-0.8	-3.43	2.68
<u>Bitcoin demand est.</u>										
$w_{b,t}^{(cor)}$, (%)	1.89	2.85	0.1	-4.38	9.05	2.94	2.37	0.2	-4.25	9.05
$w_{b,t}^{(vol)}$, (%)	5.56	8.57	2.9	-0.21	48.95	3.16	4.42	2.9	-0.21	34.63
$w_{b,t}^{(c+v)}$, (%)	5.01	8.08	2.9	-4.24	48.94	3.89	5.01	2.0	-4.24	34.97
$\Delta w_{b,t-1:t}^{(cor)}$	-0.00	0.61	-0.3	-5.09	6.55	0.00	0.57	1.9	-4.27	6.55
$\Delta w_{b,t-1:t}^{(vol)}$	-0.01	3.41	0.4	-37.65	30.82	0.02	2.41	3.3	-12.23	22.66
$\Delta w_{b,t-1:t}^{(c+v)}$	-0.01	3.66	-1.0	-45.63	28.74	0.02	2.51	3.0	-12.21	22.26

Table A.3: Correlation Matrix of Bitcoin Demand Estimates

The table shows a correlation matrix of the implied variables from DCC-GARCH estimates based on the post-futures period 12/18/2017 - 12/31/2020. r_m and r_b are daily log returns of S&P500 index and Bitcoin (USD), respectively. β_t and $\rho_{bm,t}$ are daily CAPM beta and conditional correlation estimates from DCC-GARCH fitting. $w_{b,t}^{(cor)}$, $w_{b,t}^{(vol)}$, and $w_{b,t}^{(c+v)}$ denote the non-speculative demands due to correlation changes, volatility ratio changes, and both correlation and volatility ratio changes, respectively. All variables are at daily frequency.

	β_t	$\rho_{bm,t}$	$\frac{SD_t(r_b)}{SD_t(r_m)}$	$w_{b,t}^{(cor)}$	$w_{b,t}^{(vol)}$	$w_{b,t}^{(c+v)}$	$\Delta w_{b,t}^{(cor)}$	$\Delta w_{b,t}^{(vol)}$	$\Delta w_{b,t}^{(c+v)}$	$\Delta w_{b,t}^{(cor)}$ $+\Delta w_{b,t}^{(vol)}$
β_t										
$\rho_{bm,t}$	0.82									
$\frac{SD_t(r_b)}{SD_t(r_m)}$	-0.06	-0.35								
$w_{b,t}^{(cor)}$	-0.79	-0.99	0.36							
$w_{b,t}^{(vol)}$	0.01	0.37	-0.65	-0.39						
$w_{b,t}^{(c+v)}$	-0.26	0.01	-0.55	-0.03	0.92					
$\Delta w_{b,t}^{(cor)}$	-0.09	-0.09	-0.03	0.09	-0.05	-0.02				
$\Delta w_{b,t}^{(vol)}$	-0.04	-0.03	-0.07	0.03	0.14	0.17	0.02			
$\Delta w_{b,t}^{(c+v)}$	-0.04	-0.04	-0.06	0.04	0.12	0.16	0.19	0.98		
$\Delta w_{b,t}^{(cor)}$ $+\Delta w_{b,t}^{(vol)}$	-0.05	-0.04	-0.07	0.04	0.13	0.16	0.19	0.98	0.99	

A.4 Uncertainty in Sequentially Estimated Correlation

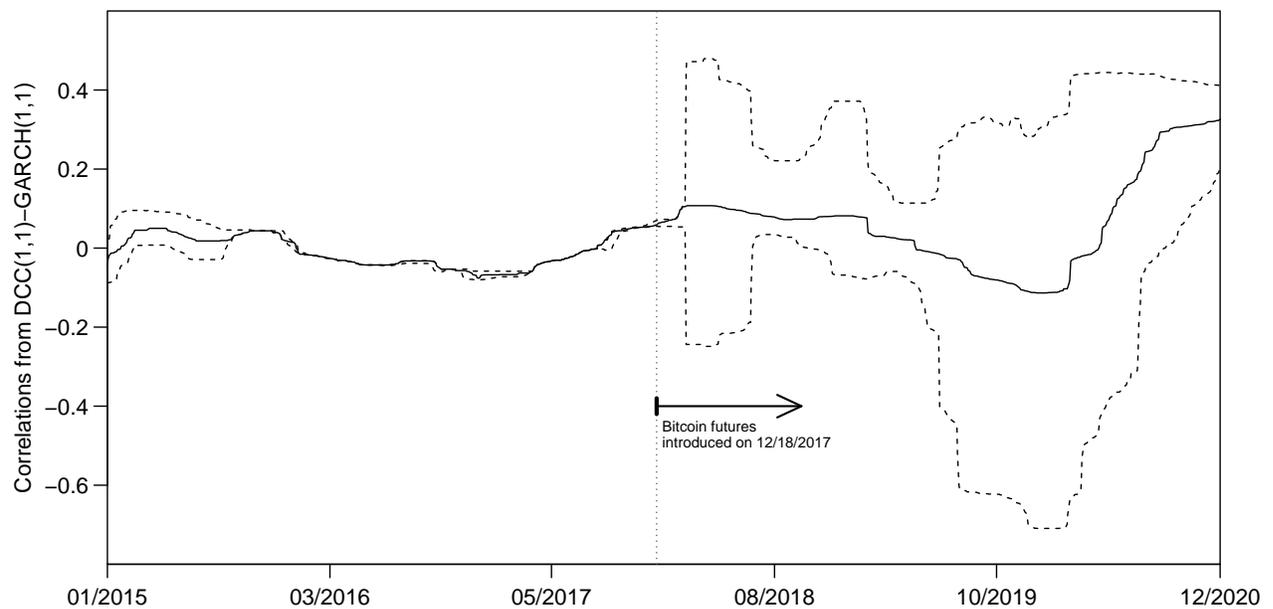
At each day, we fit a DCC(1,1)-GARCH(1,1) model using the log returns of Bitcoin and SP&500 index for the past two years. For example, on May 1, 2015, we use daily log returns data from May 1, 2013 to April 30, 2015 for estimation.

Figure A.2 shows how investors' perception on correlation dynamics changes in real-time. The dashed lines show the time-variation of the 99% intervals—the 0.5th and 99.5th percentiles—of the estimated time-series of conditional correlations $(\rho_{t-251}, \rho_{t-250}, \dots, \rho_t)$ from a past-looking window of 12 months whereas the solid line shows their medians.^{A.2} Therefore, a wide gap between dashed lines represents a huge time-variation of the conditional correlations in the past 12 months, which is identified by investors on a given day. If this rolling-window estimation approximates investors' sequential learning of the joint dynamics of Bitcoin and stocks, then the pattern in the graph implies that at least some investors must recognize the structural change in early January of 2018. Therefore, if investors' reaction to the time-varying correlation affects Bitcoin prices, the effect should appear but not prior to 2018.

^{A.2}To smooth out the graphs, we use the medians of 0.5th, 50th, and 99.5th percentiles from 21 trading days (one month) around the given date. That is, ± 10 days around a given day t .

Figure A.2: Time-Variations of Sequentially Estimated Correlation Dynamics

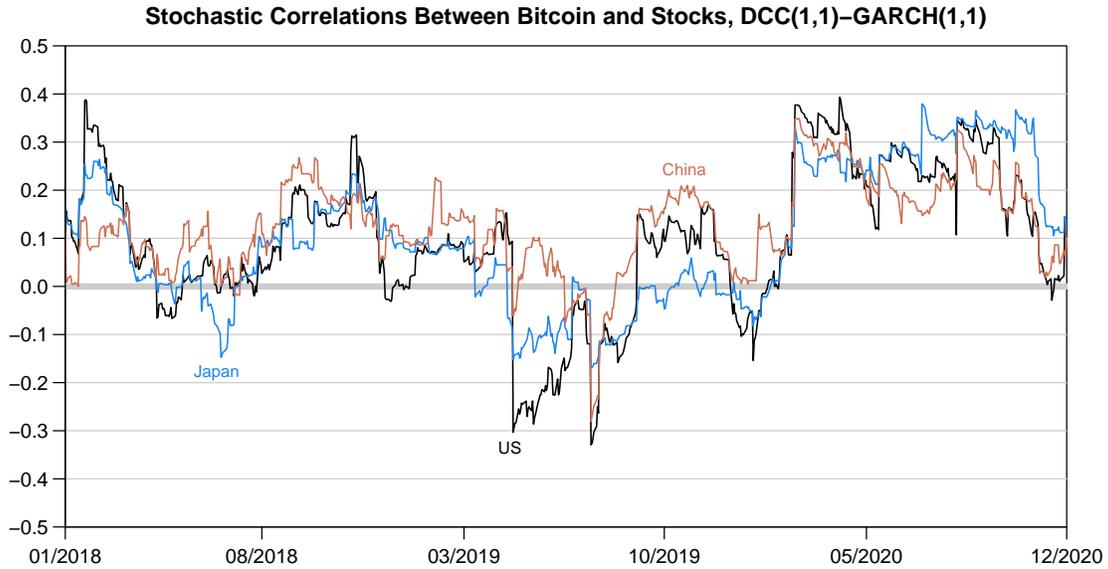
The plot illustrates how investors' perception on correlation dynamics changes in real-time. The dashed lines show the time-variation of the 99% intervals—the 0.5th and 99.5th percentiles—of the estimated time-series of conditional correlations $(\rho_{t-251}, \rho_{t-250}, \dots, \rho_t)$ from a past-looking window of 12 months whereas the solid line shows their medians. To smooth out the graphs, we use the medians of 0.5th, 50th, and 99.5th percentiles from 21 trading days (one month) around the given date. That is, ± 10 days around a given day t . Therefore, a wide gap between dashed lines implies that investors on a given day notice a huge time-variation in their conditional correlation estimates from DCC(1,1)-GARCH(1,1) fitting with the past 12 months of data.



A.5 Time-Varying Correlations of Bitcoin with Global Equities

Figure A.3: Time-Varying Correlations Between Bitcoin and S&P500 Index

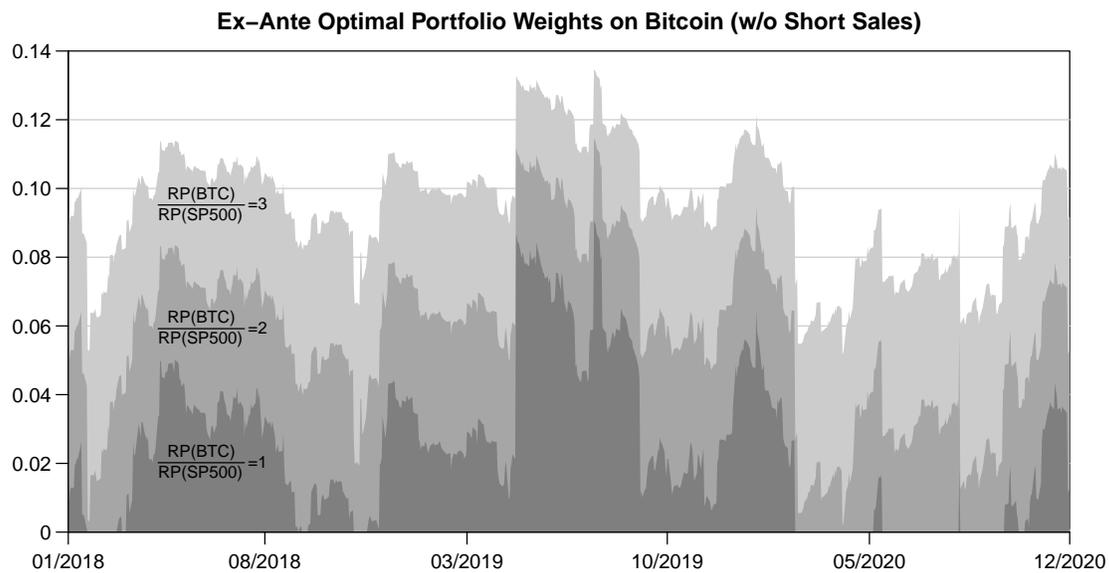
The figure shows the time-varying correlation estimates from DCC(1,1)-GARCH(1,1) models with daily log returns of Bitcoin with S&P500 Index (US, black line), Shanghai Composite (China, red line), and Nikkei 225 (Japan, blue line).



A.6 Optimal Portfolio Weights on Bitcoin

Figure A.4: Time-Varying Optimal Portfolio Weights on Bitcoin

The figure displays estimated time-varying portfolio demand for Bitcoin due to conditional correlation changes, in terms of optimal portfolio weights on Bitcoin with short sale constraints.



A.7 Diversification Benefits of Bitcoin

First, we compute the optimal weights on Bitcoin by mixing Bitcoin and a S&P500 index fund to maximize Sharpe ratio. Then Figure A.5 Panel (a) shows that the proportion of positive (daily) optimal weights $w_{b,t}^{(c+v)}$ on Bitcoin, for each value of $\bar{\mu}^*$. That is, each point on the graph is based on a unique value of $\bar{\mu}^*$ in the horizontal axis. The proportion starts with 84% at $\bar{\mu}^* = 1$ (the most conservative estimate on risk premium in Bitcoin) and gradually increases with $\bar{\mu}^*$ until it reaches almost 100% at $\bar{\mu}^* = 2$. The median optimal weight $w_{b,t}^{(c+v)}$ at $\bar{\mu}^* = 1$ is about 2.5% and increases with $\bar{\mu}^*$. Positive optimal weights on Bitcoin imply that Sharpe ratio is improved and the efficient frontier moves up by including Bitcoin in the portfolio.

Figure A.5 Panel (b) shows the median and (25th, 75th) percentiles of daily ex-ante Sharpe-ratio-improvement metric for the post-futures period with $\bar{\mu}^*$ ranging from 1 to 3.26.

$$(\text{Sharpe ratio improvement}) = \frac{SR_t(r_{p,t+1})}{SR_t(r_{m,t+1})} - 1,$$

The bottom 25th percentile of this metric is positive even at $\bar{\mu}^* = 1$ and increases with $\bar{\mu}^*$. Unlike the existing literature, we compute ex-ante Sharpe ratios rather than ex-post versions based on realized excess returns. We find that Sharpe ratios computed by ex-post returns are unreliable because of Bitcoin's dramatic price swings; by contrast, our results are robust to different choices on sample period

Figure A.5: Optimal Weights on Bitcoin in Passive Portfolios

Panel (a) shows the proportion of *positive* optimal weights $w_{b,t}$ on Bitcoin among 765 days in the post-futures sample for different levels of $\bar{\mu}^* = RP(r_{b,t})/RP(r_{m,t})$, that is, the ratio of Bitcoin' risk premium over S&P500's on a horizontal axis. For Panel (b), we first define 'ex-ante Sharpe-ratio improvement by including Bitcoin' as follows:

$$(\text{Sharpe ratio improvement}) = \frac{SR_t(r_{p,t+1})}{SR_t(r_{m,t+1})} - 1,$$

where $r_{p,t+1}$ is a return of a portfolio that optimally mixes Bitcoin and S&P500 with a short-sale constraint while $r_{m,t+1}$ is a return of S&P500. Then Panel (b) repeats Panel (a) but draws the median (solid line) and (25th, 75th) percentiles (dashed lines) of 'ex-ante Sharpe ratio improvement by including Bitcoin', among 765 days in the post-futures sample.

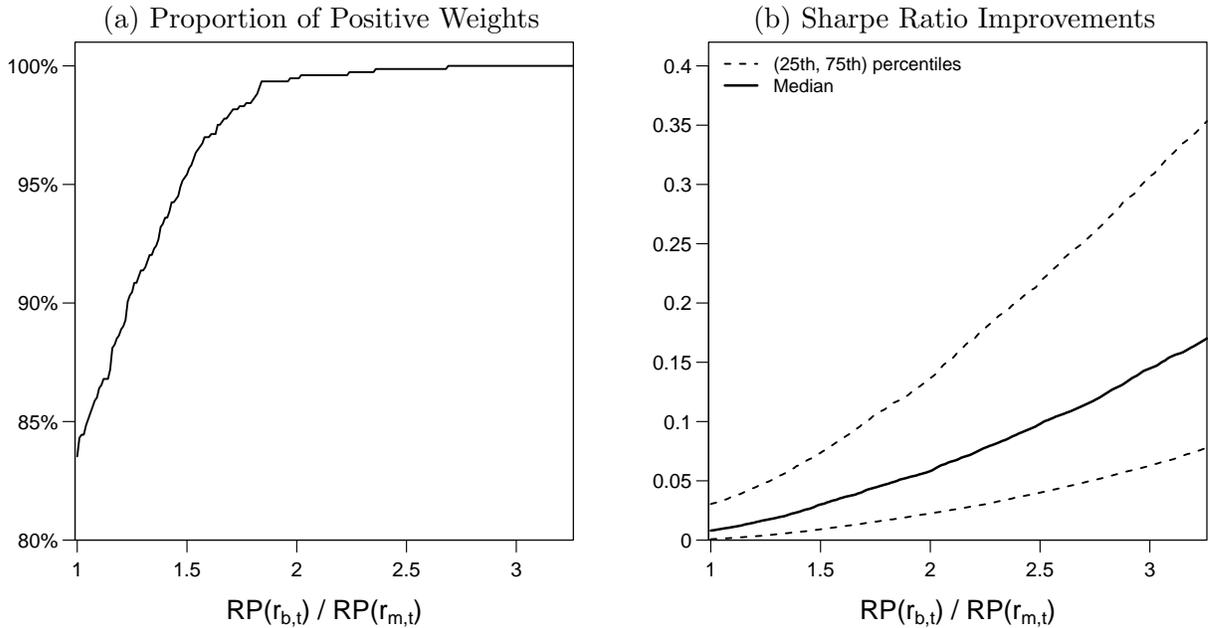
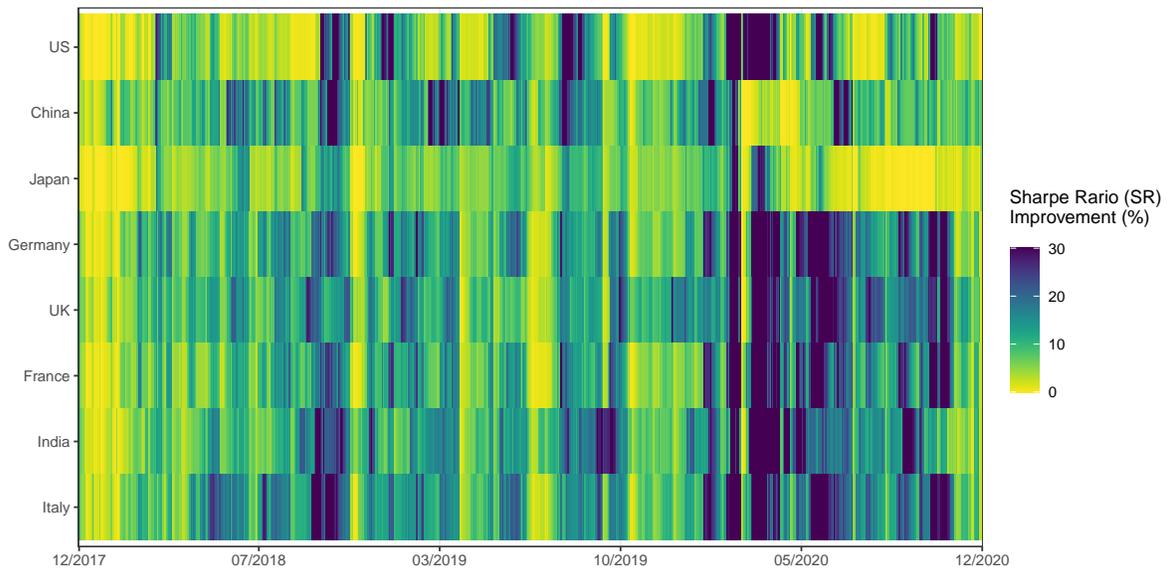


Figure A.6: Sharpe Ratio Improvements by Bitcoin in Global Equity Markets.

The figure shows the conditional Sharpe ratio improvements by adding Bitcoin in global equities.



A.8 Robustness Tests

A.8.1 Bitcoin Demand Construction

We derive non-speculative demand for Bitcoin in Section 3 relying on a few approximations. First, we use $\mu_t^* = \frac{\mu_{b,t}}{\mu_{m,t}} = 1$ for simplicity but it can be actually any value between 1 and $\frac{\sigma_{b,t}}{\sigma_{m,t}} = 3.26$ from the inequality (A.4). Second, computing correlation-related conditional demand $w_{b,(t-1):t}^{(cor)}$ requires $\bar{\sigma}^*$ which we replace by the sample median of time-varying volatility ratios from the DCC(1,1)-GARCH(1,1) estimation. Similarly, computing volatility-ratio-related conditional demand $w_{b,(t-1):t}^{(vol)}$ needs $\bar{\rho}^*$ which we replace by the sample median of time-varying correlation coefficients from the DCC(1,1)-GARCH(1,1) estimation.

Figure A.7 Panel (a) displays how the Newey-West t-statistics of the coefficients b_1 and b_2 in the predictive regression $r_{b,t+1} = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + b_2 \Delta w_{b,(t-1):t}^{(vol)} + e_{t+1}$ will vary with different choices on μ_t^* . The Newey-West t-statistics of the coefficient b_1 are stable around 4 even with different μ_t^* values ranging from 0 to 3.26 whereas the Newey-West t-statistics of the coefficient b_2 are always below 1 and mostly close to zero. Additionally, we repeat the same procedure the using different values for $\bar{\sigma}^*$ and $\bar{\rho}^*$. The Newey-West t-statistics of the coefficient b_1 are almost invariant even if we replace $\bar{\sigma}^*$ by 25% or 75% percentiles of σ_t from the DCC-GARCH estimation. On the other hand, the Newey-West t-statistics of the coefficient b_2 are still lower than 1 and mostly close to zero even if we replace $\bar{\rho}^*$ by -0.5 or $+0.5$. Hence, our results are robust for different values of $\bar{\sigma}^*$ and $\bar{\rho}^*$. This is not surprising given that our predictor is the change in demand, not the level.

Figure A.7 Panel (b) repeats the main results in Panel (a) but using different DCC-GARCH models and distribution assumptions. We estimate DCC(1,1)-GARCH(1,1), asymmetric DCC(1,1)-GARCH(1,1) (namely, aDCC-GARCH), and DCC(1,1)-EGARCH(1,1). For distribution assumptions, we explore t-distribution in addition to normal distribution. Similarly to Panel (a), Newey-West t-statistics of the coefficient b_1 are stable around 4 while those of b_2 are less than 1 and mostly close to zero. Therefore, our main findings are robust to various DCC-GARCH and distribution assumptions.^{A.3}

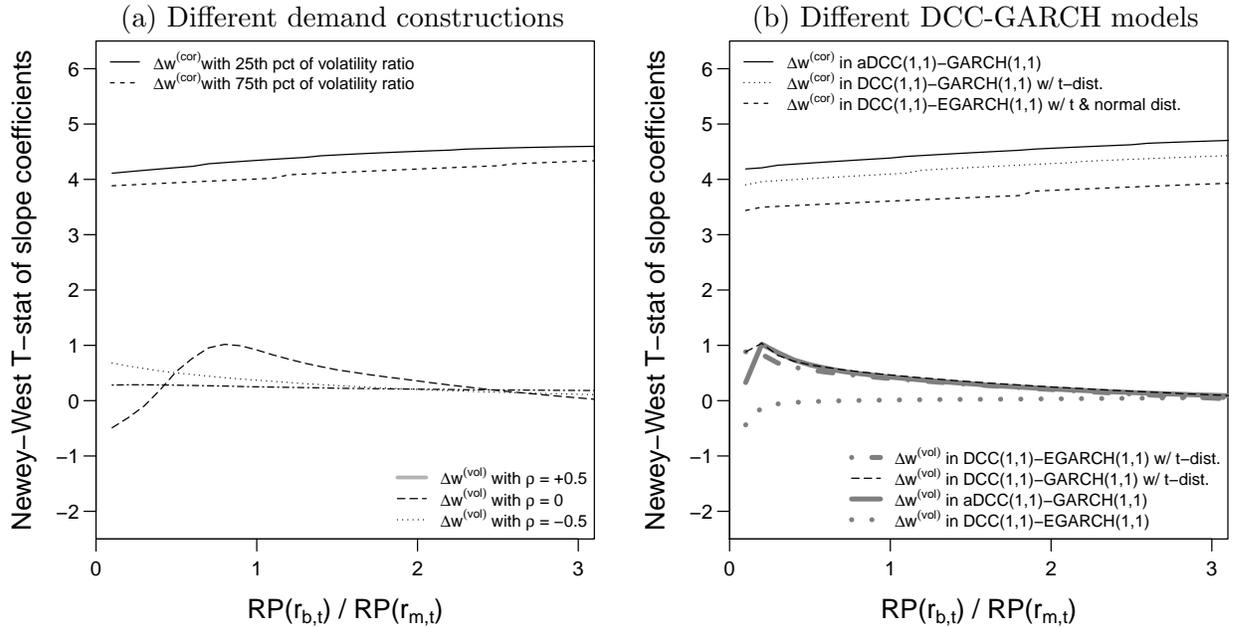
^{A.3}We do not report here, but adding autoregressive components in the mean function of DCC-GARCH model strengthen our results.

Figure A.7: Robustness of Return Predictability

This figure displays how the Newey-West t-statistics of the coefficients b_1 and b_2 in the following regression vary with different choices on μ_t^* in Bitcoin demand construction in Section 3.

$$r_{b,t+1} = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + b_2 \Delta w_{b,(t-1):t}^{(vol)} + \varepsilon_{t+1}$$

The horizontal axis $\mu_t^* = \frac{\mu_{b,t}}{\mu_{m,t}}$ ranges from 0 to 3.22, consistent with the inequality (A.4). Then different lines in Panel (a) show Newey-West t-statistics of b_1 and b_2 with different values for $\bar{\sigma}^*$ and $\bar{\rho}^*$ at demand construction. We replace $\bar{\sigma}^*$ by 25% or 75% percentiles of σ_t from DCC-GARCH estimation. Also we replace $\bar{\rho}^*$ by -0.5 , 0 , or $+0.5$. Panel (b) repeats the main result in Panel (a) but using different DCC-GARCH specifications and distribution assumptions such as t-distribution instead of Normal distribution.



A.8.2 Quantile Regression

We repeat our main predictive regression with quantile objective functions to confirm robustness against non-normality and extreme observations. The quantile regression satisfies the following:

$$Q(\tau) = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + b_2 \Delta w_{b,(t-1):t}^{(vol)}$$

where $Q(\tau)$ is $(\tau \times 100)^{th}$ percentile of Bitcoin's daily log returns $r_{b,t+1}$. $\Delta w_{b,(t-1):t}^{(cor)}$ and $\Delta w_{b,(t-1):t}^{(vol)}$ refer to non-speculative demand changes due to correlation and volatility ratio changes from $t - 1$ to t , respectively. When $\tau = 0.5$, we have least absolute deviations (LAD) regressions. Figure A.8 shows how coefficient estimates b_1 and b_2 with their 95% confidence intervals varies depending on the choice of the quantile criteria τ in Panel (a) and (b), respectively.

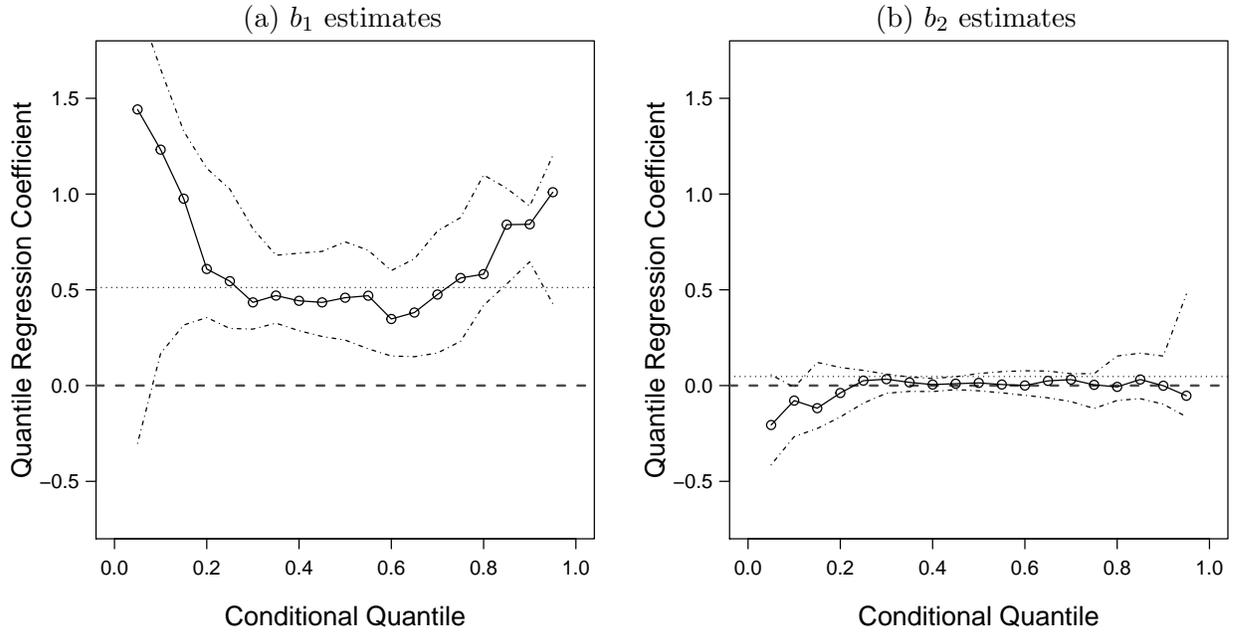
In Figure A.8, the coefficient on $\Delta w_{b,(t-1):t}^{(cor)}$, i.e., b_1 , varies around the previous OLS estimate 0.5 while keeping its statistical significance for the most part, except $\tau < 0.1$. On the other hand, the coefficients on $\Delta w_{b,(t-1):t}^{(vol)}$, i.e., b_2 , are rather flat around zero and insignificant over all quantiles. The quantile regression results confirm robustness of Bitcoin return predictability.

Figure A.8: Quantile Predictive Regressions

This figure shows coefficients (solid lines with circle markers) and their 95% confidence intervals (dash-dotted lines) from the quantile regressions satisfying the following:

$$Q(\tau) = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + b_2 \Delta w_{b,(t-1):t}^{(vol)}$$

where $Q(\tau)$ is $(\tau \times 100)^{th}$ percentile of Bitcoin's daily log returns $r_{b,t+1}$. $\Delta w_{b,(t-1):t}^{(cor)}$ and $\Delta w_{b,(t-1):t}^{(vol)}$ refer to non-speculative demand changes due to correlation and volatility ratio changes from $t - 1$ to t , respectively. When $\tau = 0.5$, we have least absolute deviations (LAD) regressions.



A.8.3 Trimming and Winsorizing

Bitcoin returns are highly volatile and fat-tailed. A few extreme observations can distort simple OLS estimates. Here, we trim or winsorize the data to see how the extreme values affect our results in the predictive regression $r_{b,t+1} = b_0 + b_1 \Delta w_{b,(t-1):t}^{(cor)} + \varepsilon_{t+1}$. For trimming, we omit observations whenever $r_{b,t+1}$ is outside the interval $(c, 100 - c)$ percentiles. For winsorizing, we replace both dependent and independent variables outside their own intervals $(c, 100 - c)$ percentiles by their nearest boundary values. Then we fit the predictive regression and visualize the results for each cutoff point c (ranging from 0 to 5 percentiles) in Figure A.9 and A.10, for the post-futures period with and without the COVID-19 era samples. In Figure A.9, we observe that the coefficient b_1 from trimmed data is around 0.5 and statistically significant even if we omit total 10% of the sample at tails. In Figure A.10, the coefficient b_1 from the winsorized data even increases with the cutoff c .

Figure A.9: Predictive Regressions with Trimmed Data

The figure shows how different trimming schemes change OLS estimates on b_1 (solid lines with circle markers) and associated 95% confidence intervals (dash-dotted lines) from Newey-West standard errors, for two different sample periods.

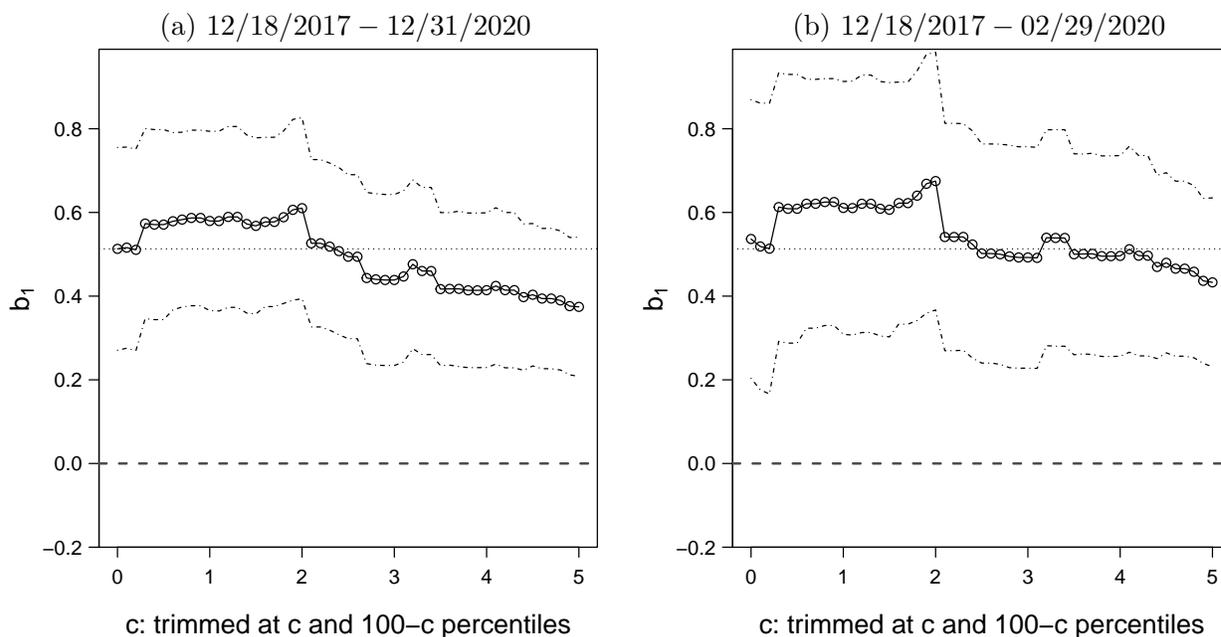
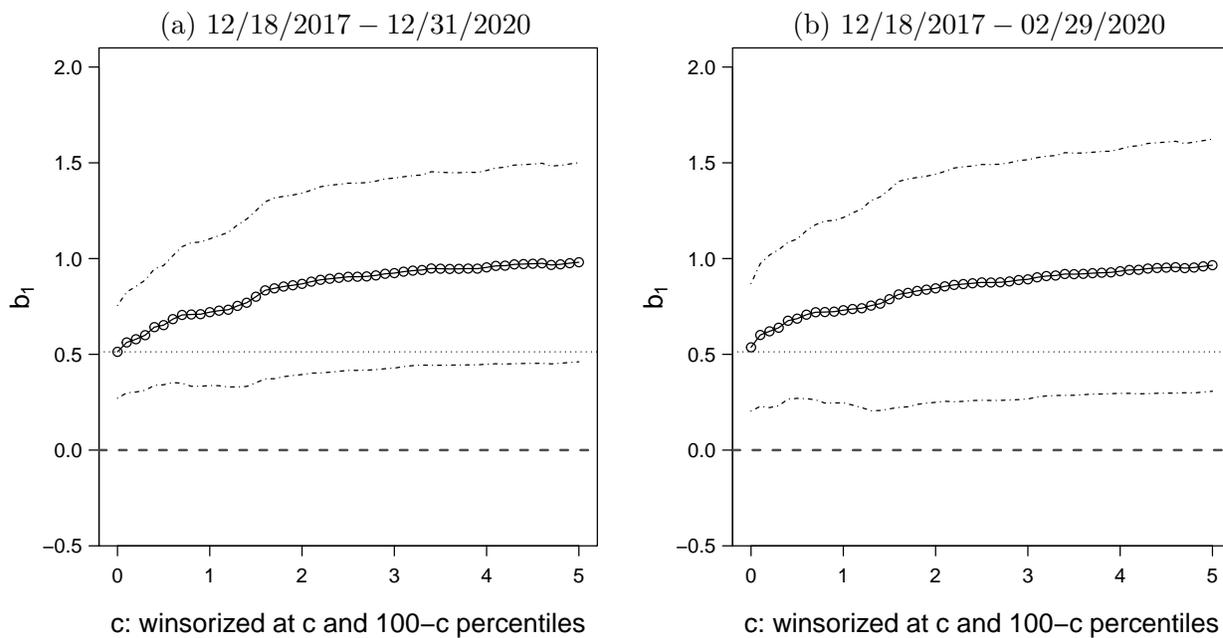


Figure A.10: Predictive Regressions with Winsorized Data

The figure shows how different winsorizing schemes change OLS estimates on b_1 (solid lines with circle markers) and associated 95% confidence intervals (dash-dotted lines) from Newey-West standard errors, for two different sample periods.



A.8.4 Out-of-Sample Binary Prediction - Confusion Matrix

As another robustness check, we provide a confusion matrix and related tests for a binary classifier (price up/down) rather than regressions. With the intuition behind our Bitcoin demand proxy, we create a naive classification rule: predict ‘up’ if $\Delta w_{b,(t-1):t}^{(cor)} > 0$ and ‘down’ otherwise. We re-estimate the DCC(1,1)-GARCH(1,1) every trading day to construct $\Delta w_{b,(t-1):t}^{(cor)}$ in a truly out-of-sample fashion. Note that classifier is not trained from forecast errors for better return predictability. We intentionally avoid optimizing the prediction and construct a conservative classifier to test robustness of our return predictability. We only use DCC-GARCH estimation results to construct $\Delta w_{b,(t-1):t}^{(cor)}$. The evaluation sample is from 01/02/2019 to 12/31/2020 and we sequentially increase the training sample period.

In Table A.4 Panel A, the confusion matrix shows that Sensitivity = 0.615, Specificity = 0.496, Positive Predicted Value (PPV) = 0.589, Negative Predictive Value (NPV) = 0.523, and Accuracy (ACC) = 0.560, which indicates that the binary prediction is informative. If the classifier is purely random, both ‘Sensitivity + Specificity’ and ‘PPV + NPV’ should be approximately one in population. Panel B shows that the 95% confident intervals for both ‘Sensitivity + Specificity’ and ‘PPV + NPV’ have the lower bound at 1.025, which is greater than 1, also suggesting that the classifier is significantly informative.

We perform three formal tests for the binary classifier: Fisher’s Exact test, Chi-square test, and Wilcoxon Rank-sum Test.^{A.4} In all three tests, no p-value exceeds 5%, suggesting that the classifier well predicts the Bitcoin price movement direction at 5% significance level. As a sensitivity check, we try different cutoff values for the classifier between -0.002 and 0.002, which are the 15th and the 80th percentiles of $\Delta w_{b,(t-1):t}^{(cor)}$, as shown in Figure A.11 Panel (a). We also try different ending dates of the evaluation period, as in Panel (b). The p-values of all three tests in both panels fluctuate but do not exceed 5%. All the evidence with this binary classifier supports Bitcoin return predictability from $\Delta w_{b,(t-1):t}^{(cor)}$.

^{A.4}Fisher’s Exact test and Chi-square test share the null hypothesis H_0 : true (predicted) positives and true (predicted) negatives are equally likely to be predicted (true) positives. Wilcoxon Rank-sum Test has the null hypothesis H_0 : the median return of the predicted positives is equal to the median return of the predicted negatives.

Table A.4: Evaluation of Out-of-Sample Prediction

This table evaluates the performance of out-of-sample binary prediction using the predictive period from 01/02/2019 to 12/31/2020. Panel A shows a confusion matrix based on predicted binary outcomes: positive or negative Bitcoin returns. Sensitivity, or True Positive Rate (TPR), is calculated as the number of true positives divided by the number of real positives. Specificity, or True Negative Rate (TNR), is computed as the number of true negatives over the number of real negatives. PPV stands for Positive Predicted Value, calculated as the number of true positives divided by the number of predicted positives. NPV, or Negative Predictive Value, is computed as the number of true negatives divided by the number of predicted negatives. Accuracy is equal to the number of correctly predicted as a fraction of the total number of the sample observations. Panel B reports the 95% confident intervals for ‘Sensitivity + Specificity’ and ‘PPV + NPV’, computed analytically as follows.

$$TPR + TNR \pm 1.96 \times \sqrt{(TPR \times (1 - TPR)/n_1 + TNR \times (1 - TNR)/n_2)}$$

where n_1 is the number of real positives and n_2 is the number of real negatives. Similarly,

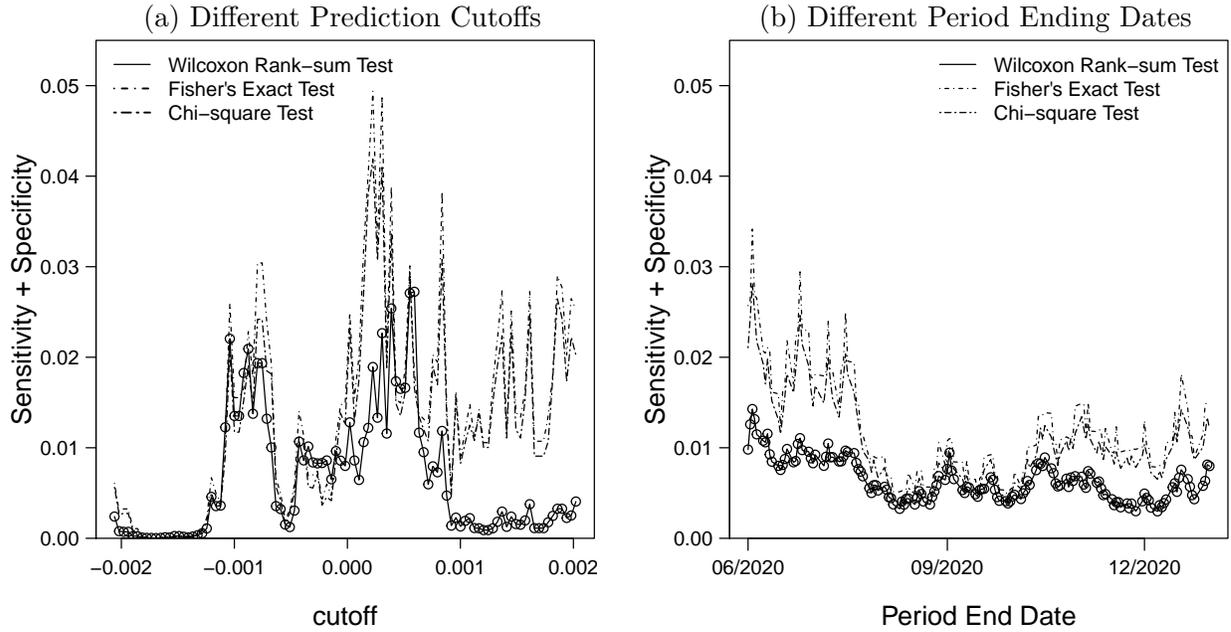
$$PPV + NPV \pm 1.96 \times \sqrt{(PPV \times (1 - PPV)/n_3 + NPV \times (1 - NPV)/n_4)}$$

where n_3 is the number of predicted positives and n_4 is the number of predicted negatives. Bootstrapping methods produce practically the same results.

Panel A: Confusion Matrix			
	Real Positive	Real Negative	
Predicted Positive	168	117	PPV = 0.589
Predicted Negative	105	115	NPV = 0.523
	Sensitivity = 0.615	Specificity = 0.496	Accuracy = 0.560
Panel B: 95% Confidence Intervals			
	Lower Bound	Upper Bound	
Sensitivity + Specificity	1.0246	1.1975	
PPV + NPV	1.0249	1.1995	

Figure A.11: P-Values of Three Tests for Confusion Table

This figure displays how the p-values of Wilcoxon Rank-sum tests, Fisher's Exact tests, and Chi-square tests for the summation of sensitivity and specificity greater than one vary with respect to the different prediction cutoffs in Panel (a) and ending dates of evaluation sample in Panel (b). Low p-values implies high levels of informativeness of the classifier.



A.9 Placebo Tests

The first three columns (1)-(3) repeat the predictive regressions in Table 6 for the full sample, splitting the independent variables into two pieces so that the coefficient on $\Delta w_{b,(t-1):t}^{(cor)}$ or $\Delta w_{b,(t-1):t}^{(vol)}$ can take different values for two subsample periods. The next three columns (4)-(6) are the placebo tests to predict only for the pre-futures sample using the illusory Bitcoin demand proxies estimated from the full sample. Finally, the columns (7)-(9) repeat the columns (1) and (2) in Table 6 for the post-futures sample as a benchmark.

Table A.5 confirms our hypothesis. The coefficient on $\Delta w_{b,(t-1):t}^{(cor)}$ for the pre-futures sample is small and insignificant, ranging from -0.07 to -0.10 with t-statistics around -0.6. Yet, the same coefficient for the post-futures sample is about 0.5 and statistically significant with t-statistics around 4. On the other hand, The coefficient on $\Delta w_{b,(t-1):t}^{(vol)}$ is small and insignificant regardless of sample periods, ranging from -0.21 to 0.06.

Table A.5: Predictive Regression in Sub-Periods

This table repeats columns (1) and (2) of Table 6 for three different sample periods: full (01/01/2015 to 12/31/2020), pre-futures (01/01/2015 to 12/17/2017) and post-futures (12/18/2017 to 12/31/2020). The reported numbers are coefficients and associated Newey-West t-statistics (in parenthesis). $I^{(1)}$ is an indicator function whose value takes one for the pre-futures sample and zero otherwise whereas $I^{(2)}$ is similar but for the post-futures sample. For columns (1)-(6), $\Delta w_{b,(t-1):t}^{(cor)}$ and $\Delta w_{b,(t-1):t}^{(vol)}$ are estimated using the full sample while using only post-futures sample for columns (7)-(9).

Predictor	Full			Pre-futures			Post-futures		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$\Delta w_{b,(t-1):t}^{(cor)} I^{(1)}$	-0.10 (-0.72)		-0.07 (-0.51)	-0.10 (-0.71)		-0.07 (-0.52)			
$\Delta w_{b,(t-1):t}^{(cor)} I^{(2)}$	0.52 (4.35)		0.52 (4.11)				0.51 (4.15)		0.51 (4.09)
$\Delta w_{b,(t-1):t}^{(vol)} I^{(1)}$		-0.21 (-1.21)	-0.19 (-0.92)		-0.21 (-0.97)	-0.19 (-0.95)			
$\Delta w_{b,(t-1):t}^{(vol)} I^{(2)}$		0.04 (0.20)	0.02 (0.17)					0.06 (0.42)	0.05 (0.42)
R^2 (%)	0.68	0.10	0.77	0.06	0.22	0.24	1.15	0.01	1.16