Mean Variance Portfolio Choice with Uncertain Mean and Variance Covariance Matrix

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Abstract

This paper examines why in the real world investors hold few stocks and ignore diversification benefit from a large number of stocks. Several literatures explain that ambiguities toward assets is the reason of poor diversification. Specifically, Boyle et al (2010) show how return ambiguity causes less participation in unfamiliar asset. Since in the real world an investor has both return and variance ambiguity, we develop a new model that incorporates ambiguities of both parameters and make a comparison with other models. We find that return and variance ambiguities cause a bias investment toward familiar asset, participation in only familiar asset and non-participation in risky assets. Our result explains why employees have a bias investment in their own-company, why investors have poor diversification in international stocks and why non-participation in stocks occurs.
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1. Introduction

If diversification among a large number of assets gives the highest risk-adjusted return as the theory suggests, why would a large number of investors choose to allocate their capitals in few assets? This paper will explain this empirical situation by assets’ return and variance ambiguity.

There are two schools of thought regarding asset allocation: Keynes and Markowitz. Markowitz’s theory (1952, p. 77) is well-known in the literature, showing that diversification gives a mean-variance investor the highest risk-adjusted return. His theory recommends diversifying across a large number of assets and forming the optimal portfolio. Then, the investor only invests in the optimal portfolio and risk-free asset. However, empirical evidence is against Markowitz’s and supports Keynes’s thought. Keynes’s theory states that the investor should only participate in the small number of assets which he is more familiar with.

Boyle et al (2010) come up with their model to compromise these two thoughts. They show that, by adding return ambiguity to the classical Markowitz’s mean variance model, their new model gives the result similar to the empirical evidence that investors hold a small number of assets. However, authors have an assumption that the investors do not have variance ambiguity, which is not true and realistic. Wang (2012) addresses variance uncertainty by adding variance ambiguity into the classical model. However, he does not add return ambiguity. Thus, it has a similar mistake; his model is unrealistic since the investors do not only have variance but also return ambiguity.

Therefore, in this paper we develop a new model that incorporates both return and variance ambiguities into classical mean variance model. Since an investor has both return and
variance ambiguities, our model is more accurate than the model by Boyle et al (2010) and Wang (2012). We explore the effect of return and variance ambiguities on asset allocation of familiar and unfamiliar assets and compare our model with the models from the previous literature.

Our model shows more reduction in the holding of the ambiguity-asset than the return-ambiguity model alone and variance-ambiguity model alone. Under-diversification and concentration of wealth in a more familiar asset is observed. When ambiguities of assets are sufficiently high, the investor does not participate in stock market. Thus, we observe non-participation.

Our model explains empirical observation that investors under-diversify and put a significant amount of wealth in their own company’s stocks. In addition, it explains why many people invest only in risk-free asset and don’t participate in risky assets. It also explains home-bias in which an investor puts more weight in his home country’s stocks and less weight on international assets regardless of diversification benefits. Our model gives a reason to the empirical puzzle why an investor ignores diversification benefit, holding a small number of assets rather than a large number of asset; he follows Keynes’s view rather than Markowitz’s.

This paper is organized as follow; 2. Literature review 3. Methodology 4. Result and Interpretation 5. Data and a numerical example 6. Empirical evidence 7. Conclusion

2. Literature review

Markowitz’s (1952, 1959) theory states that an optimal mean variance investor should allocate his fund into the optimal portfolio which is diversified across a large number of assets and a risk-free asset to obtain the highest risk-adjusted return. The optimal portfolio is the efficient portfolio that gives the optimal Sharpe ratio. Efficient portfolios have either the highest
return given variance or lowest variance given return from diversification. Utility and risk aversion of an investor determine the weight between the optimal portfolio and risk-free asset. The model using Markowitz’s theory is called “classical mean variance model” which has a good reputation in financial sectors.

However, there are various empirical studies showing a contrasting behavior of an investor. Instead of putting capital in the optimal portfolio which is diversified across a large number of assets, many investors allocate their capitals in few assets, showing poor diversification. Blume and Friend (1975) find that an investor allocate his wealth to one to two stocks by using data from return on income-tax. In addition, Kelly (1995) finds that a median U.S. investor holds one stock, usually his own company’s using data from the Survey of Consumer Finances (SCF). Polkovnichenko (2005) finds that investors who allocate his wealth to individual stocks hold median stocks of two in 1983 and three in 2001, using data from SCF and they invest in the companies that they work for. Barber and Odean (2000) find inefficient diversification, using data from U.S. brokerage. Calvat, Campbell and Sodini (2007) find that Swedish households have poor diversification, using Swedish government record.

In addition, many studies find that investors have a bias toward familiar asset. Huberman (2001) shows many evidences that investor puts relatively more weight in familiar assets than unfamiliar assets. Masa and Simonov (2006) find that investors hold a great share of stocks of companies which is located close to them. In addition, home bias is evidenced in various studies. French and Poterba (1990), Cooper and Kaplanis (1994) observe that investors hold home equity instead of diversifying across international equities. Coval and Moskovitz (1999) find that small and large investors have a bias toward local stocks that are located close to them. Home bias is further observed in various countries. Grinblatt and Keloharju (2001) find Finnish investors
having bias in their home equity. Similarly, Massa and Simonov (2006) find the bias among Swedish investors. Feng and Seasholes (2004) find that investors in China have a bias in their local companies.

There are several papers that explain why an investor holds a small number of assets. Brennan (1975) explains that transaction cost makes an investment in large number of assets unappealing. Merton (1987) explains that investors are only aware of few assets. There are many other studies such as Kraus and Litzenberger (1976), explaining that skewness causes less diversification. Cohen (2009) asserts that employees invest in the companies they work because of their loyalty. Shefrin and Statman (2000) give the reason that an investor cares for downside protection and upside gain more than diversification. Liu (2009) gives the reason of an insurance as a result of margin and short-sale constraints. Odean (1999) states that overconfidence causes poor diversification.

Similar to Boyle et al (2010), we believe that information advantage by Van Nieuwerburgh and Veldkamp (2008) does not fully explain poor diversification. They create a model to explain concentration investment in certain assets, biasness toward own-company stock and home bias puzzle, using increasing return to scale. However, their model depends on several unrealistic conditions. The first condition is that expected return of familiar asset must be higher than other assets. Second, the model requires a large number of assets as inputs and is not flexible enough to explain a simple situation with few assets. Third, the model could not explain non-participation in risky assets. Lastly, the model implies that a portfolio of few assets must outperform a portfolio with diversification, which is not always true.

Although there are many reasons to explain the poor diversification, many studies find that ambiguity is one important factor. Garlappi, Upal and Wang (2007) add return ambiguity
into their Multi-prior model using Knight’s (1921) concepts of uncertainty. They claim that in the real world, estimation of return is difficult and contains estimation error. The classical mean variance model ignores this error and assumes that an investor could estimate the return with precision. They find that their model is better than the classical model as the optimal portfolio weight does not have extreme value and is more stable over time, showing the improved result out of sample.

Later, Boyle et al (2010) use similar technique to explore the effect of ambiguity and compare the trade-off between Keynes’s and Markowitz’s thought of portfolio construction. They explore the case when an investor has ambiguity in return but has perfect estimation of variance. They define familiar asset as the asset having relatively lower ambiguity and unfamiliar asset as the asset having relatively higher ambiguity. They use confidence interval to represent return ambiguity, using the work of Gillboa and Schmeidler (1989) and Garlappi, Upal and Wang (2007). Then they minimize “the objective function” or the optimal portfolio function over expected return. Their model could explain a bias toward familiar asset. In addition, they found sole investment in familiar asset and non-participation in risky assets in some cases. However, their model is not accurate as in reality investors do not have only return ambiguity but also are uncertain about variance.

Using similar method as Garlappi, Uppal and Wang (2007), Wang (2012) explores the effect of variance ambiguity. He uses confidence interval for variance, Chi-squared distribution, to add the ambiguity into his model. Then he makes a comparison between return ambiguity model and variance ambiguity model. However, his model is unrealistic as an investor has both return and variance ambiguity, not just variance ambiguity. Moreover, he assumes that
covariance could be perfectly estimated which is not true as an investor is uncertain about volatility and variance.

3. Methodology

We create a new model that incorporates return and variance ambiguities and then compare with other models. In this section, methodology of our model and other three models are explained. Models are as follow:

1. Model 1: The classical mean variance model
2. Model 2: The mean variance model with return ambiguity only
3. Model 3: The mean variance model with variance ambiguity only
4. Model 4: The mean variance model with both mean and variance ambiguity

Model 1

Markowitz (1952, 1959) and Sharp (1970) show that optimal portfolio could be obtained by maximizing the classical model objective function.

$$\max_{\pi} \pi' \hat{\mu} - \frac{\gamma}{2} \pi' \Sigma \pi$$

Where $\pi$ is a vector of weights. $\gamma$ is a risk aversion parameter. $\hat{\mu}$ is a vector of excess return. $\Sigma$ is variance covariance matrix.

In this paper, we have short-selling constraint because in the real world many investors cannot do short-selling due to many obstacles such as regulations, transaction cost etc. Thus weights of assets are $\pi \geq 0$. 
Thus, our optimization problem is

$$\max_{\pi} \pi' \hat{\mu} - \frac{\gamma}{2} \pi' \Sigma \pi$$

Subject to

$$\pi \geq 0$$

**Model 2**

Garlappi, Uppal and Wang (2007) extended the classical model by adding return ambiguity and called it “multi-prior model” using the notion by Bewley (1988) that confidence interval represents uncertainty. The wider is the confidence interval, the higher is the uncertainty. Excess return $\hat{\mu}_t$ with large sample size has an asymptotically normal distribution. Confidence interval is expressed as

$$P \left\{ \hat{\mu}_t - Z_{\alpha/2} \frac{\hat{\sigma}_t}{\sqrt{T}} \leq \mu_i \leq \hat{\mu}_t + Z_{\alpha/2} \frac{\hat{\sigma}_t}{\sqrt{T}} \right\} = 1 - \alpha = \theta$$

Where $T$ is total number of observations, $\theta$ is confidence level, $\alpha$ is significant level, $Z$ is Z-score.

When confidence level increases, confidence interval widens. The widening of confidence interval implies that the true excess return is estimated with less accuracy, showing higher ambiguity. Therefore, in this paper, confidence level $\theta$ is used to present ambiguity. The higher $\theta$ leads to higher ambiguity, where $\theta$ is a positive value from 0 to 1. For instance, $\theta = 0.3$ corresponds to 30% confidence level. Thus, in the rest of the paper, we refer $\theta$ to ambiguity.
Moreover, investors are assumed to be ambiguity averse. Ambiguity averse of an investor contributes to additional minimization over expected return (Gilboa and Schmeidler, 1989). Garlappi, Uppal and Wang (2007) shows that the objective function becomes

$$\max_\pi \min_\mu \pi' \mu - \gamma \frac{1}{2} \pi' \Sigma \pi$$

Subject to

$$\hat{\mu}_i - \frac{Z_{\alpha/2}}{\sigma_i \sqrt{T}} \leq \mu_i \leq \hat{\mu}_i + \frac{Z_{\alpha/2}}{\sigma_i \sqrt{T}}$$ for $i = 1, \ldots, N$

Performing inner minimization results in

$$\max_\pi \pi' \hat{\mu}_i^{adj} - \gamma \frac{1}{2} \pi' \Sigma \pi$$

Where $\hat{\mu}_i^{adj} = \left\{ \hat{\mu}_1 - \text{sign}(\pi_1) \frac{\sigma_1}{\sqrt{T}} Z_{\alpha/2}, \ldots, \hat{\mu}_N - \text{sign}(\pi_N) \frac{\sigma_N}{\sqrt{T}} Z_{\alpha/2} \right\}$ is a vector of adjusted excess return.

Since we have short-selling constraint, signs of weights are always positive. Thus the vector of adjusted excess return becomes

$$\hat{\mu}_i^{adj} = \left\{ \hat{\mu}_1 - \frac{\sigma_1}{\sqrt{T}} Z_{\alpha/2}, \ldots, \hat{\mu}_N - \frac{\sigma_N}{\sqrt{T}} Z_{\alpha/2} \right\}$$

This shows that an ambiguity-averse investor would use the lower boundary of confidence interval as his adjusted excess return, resulting in a lower weight and weakening the position of the asset. When ambiguity $\theta$ is higher, confidence interval widens and its lower boundary is reduced. As a result, $\hat{\mu}_i^{adj}$ is lower. Thus, higher ambiguity causes a lower adjusted excess return of an asset.
Our optimization problem is

$$\max_{\pi} \pi' \hat{\mu}_{adj} - \frac{Y}{2} \pi' \Sigma \pi$$

Subject to

$$\pi \geq 0$$

**Model 3**

As return is usually more difficult to estimate than variance (Boyle et al, 2010), the model that has only variance ambiguity and ignores that of the return is therefore very unrealistic and inaccurate. However, we show the result of this variance-ambiguity model in order to explore the effect of variance ambiguity. Chi-square confidence interval of variance is used to add ambiguity to covariance matrix. The confidence interval of true variance is

$$p \left( \frac{(T - 1)\sigma_t^2}{\chi^2_{T-1,\alpha/2}} \leq \sigma_t^2 \leq \frac{(T - 1)\sigma_t^2}{\chi^2_{T-1,1-\alpha/2}} \right) = 1 - \alpha = \theta$$

Where $\chi^2$ is a Chi-square statistics, $\theta$ is confidence level.

Higher confidence level $\theta$ results in wider confidence interval. The widening of confidence interval implies that the true variance is estimated with less accuracy, showing higher ambiguity.

Similar to Model 2, confidence level $\theta$ is used as variance ambiguity.
The ambiguity-averse investor minimizes his objective function over asset volatility which is written as

$$\max_{\pi} \min_{\sigma} \pi'\mu - \frac{\gamma}{2} \pi'\Sigma\pi$$

Subject to

$$\frac{(T-1)\sigma_i^2}{\chi_{T-1, \alpha/2}^2} \leq \sigma_i^2 \leq \frac{(T-1)\sigma_i^2}{\chi_{T-1, 1-\alpha/2}^2} \quad \text{for } i = 1, \ldots, N$$

To inner-minimize the objective function, variance and covariance components in variance covariance matrix are replaced with adjusted variance covariance matrix which has adjusted variance and covariance. To see how adjustments of variance and covariance are made, let’s look at an example of two risky assets.

For two risky assets case, the objective function is

$$\max_{\pi} \min_{\sigma_1, \sigma_2} \pi_1\mu_1 + \pi_2\mu_2 - \frac{\gamma}{2} \left(\pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 + 2\pi_1\pi_2\rho \sigma_1\sigma_2\right)$$

As there is no short-selling constraint, weights are always positive, $\pi_1 \geq 0$ and $\pi_2 \geq 0$. To inner-minimize the above equation, $\sigma_1$ and $\sigma_2$ must have the highest value. Thus, the upper boundaries of confidence interval which has the highest value are used. Therefore, adjusted volatility is

$$\hat{\sigma}_i^{adj} = \sqrt{\frac{(T-1)\sigma_i^2}{\chi_{T-1, 1-\alpha/2}^2}}$$

Correlation is assumed to be estimated with infinite precision and thus sample correlation is used. When ambiguity $\theta$ is higher, confidence interval widens and its upper boundary increases,
resulting in higher adjusted volatility. Thus, higher variance ambiguity increases adjusted variance and covariance.

Our optimization problem is

$$\max_\pi \pi' \mu - \frac{\gamma}{2} \pi' \Sigma^{adj} \pi$$

Subject to

$$\pi \geq 0$$

Where $\Sigma^{adj}$ is an adjusted variance covariance matrix that contains adjusted volatilities

**Model 4**

In the real world investment, an investor has ambiguities in both return and variance. Therefore, we create a new model that include both return and variance ambiguities. Our Model 4 which has both ambiguities is more realistic and accurate than Model 2 which has only return ambiguity and Model 3 which has only variance ambiguity. The investor who is averse to both ambiguities minimizes the objective function with respect to excess return and variance as

$$\max_\pi \min_{\sigma, \mu} \pi' \mu - \frac{\gamma}{2} \pi' \Sigma \pi$$

Subject to

$$\frac{(T-1)\sigma_i^2}{X^2_{n-1, \alpha}} \leq \sigma_i^2 \leq \frac{(T-1)\sigma_i^2}{X^2_{n-1, 1-\alpha}}$$

for $i = 1, \ldots, N$

$$\hat{\mu}_i - Z_{\alpha/2} \frac{\hat{\sigma}_i}{\sqrt{T}} \leq \mu_i \leq \hat{\mu}_i + Z_{\alpha/2} \frac{\hat{\sigma}_i}{\sqrt{T}}$$

for $i = 1, \ldots, N$
Due to ambiguities of both moments, the constraint for this model is a combination of two interval constraints: mean constraint and variance constraint.

Inner-minimization is similar to that in Model 2 and Model 3. For excess return, as portfolio weight is always positive due to short-selling constraint, the lower boundary of confidence interval is used as an adjusted excess return to shrink the weight and weaken asset position. For variance, upper boundary of chi-square interval is used as adjusted volatility. This weakens the position of the asset as it has higher variance and covariance. Correlation is assumed to be estimated with infinite precision and thus sample correlation is used.

Thus our optimization problem is

$$\max_{\pi} \pi' \mu^{adj} - \frac{\gamma}{2} \pi' \Sigma^{adj} \pi$$

Subject to

$$\pi \geq 0$$

Where $\hat{\mu}^{adj} = \{\hat{\mu}_1 - \frac{\sigma_1}{\sqrt{T}} Z_{\alpha/2}, \ldots, \hat{\mu}_N - \frac{\sigma_N}{\sqrt{T}} Z_{\alpha/2}\}$ is a vector of adjusted excess return

$\Sigma^{adj}$ is an adjusted variance covariance matrix that contains adjusted volatilities.

**Two risky assets and one risk-free asset**

In this paper, we explore a simple case of two risky asset and one risk-free asset. The two risky assets are familiar and unfamiliar asset. Familiar asset is an asset that an investor is more familiar relative to unfamiliar asset, representing by lower ambiguity $\theta_F \leq \theta_U$ where $\theta_F$ is an ambiguity of familiar asset and $\theta_U$ is an ambiguity of unfamiliar asset. On the other hand,
unfamiliar asset is an asset that the investor is relatively less familiar relative to familiar asset, representing by higher ambiguity $\theta_U \geq \theta_F$. Risk-free asset is included as excess returns are used.

Similar to Boyle et al (2010), we assume two equal risky assets; familiar and unfamiliar assets have the same excess return and volatility and some correlation between them. Our assumption of equal assets is to see the effect of ambiguity more clearly as the two assets, when there are no ambiguities, have equal optimal weight $\pi_F = \pi_U$ where $\pi_F$ is the optimal weight of familiar asset and $\pi_U$ is the optimal weight of unfamiliar asset. Therefore, in Model 2, since $\theta_F \leq \theta_U$, thus $\hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj}$. Similarly in Model 3, $\sigma_U^{adj} \geq \sigma_F^{adj}$. Model 4 has both conditions $\hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj}$ and $\sigma_U^{adj} \geq \sigma_F^{adj}$.

4. Result and Interpretation

4.1 The Optimal portfolio weight

The optimal weights of familiar and unfamiliar assets are derived from constrained optimization. Their derivations are in Appendix.

Model 1

Case 1, Participation in both assets

$$\pi_F = \frac{\hat{\mu}}{\gamma \sigma^2(1 + \rho)} > 0$$

$$\pi_U = \frac{\hat{\mu}}{\gamma \sigma^2(1 + \rho)} > 0$$

Scenario: $\hat{\mu} > 0$
Case 2, Non-participation

\[
\pi_F = 0 \\
\pi_U = 0
\]

Scenario: \( \hat{\mu} \leq 0 \)

Case 1 happens when excess returns are positive. Weights of two assets are equal because both assets have equal excess return and variance. When excess returns increase, an investor allocate more wealth to both assets. Higher risk aversion, higher variance and higher correlation reduce the holdings of the assets.

Case 2 happens when excess return is negative or zero. Holding risk-free asset gives either higher return or similar return with no risk. Thus, since the investor could not short-sell, he does not hold any risky assets but only risk-free asset.

Model 2

Case 1, Participation in both assets

\[
\pi_F = \frac{\hat{\mu}_F^{adj} - \rho \hat{\mu}_U^{adj}}{\gamma \sigma^2 (1 - \rho^2)} > 0 \\
\pi_U = \frac{\hat{\mu}_U^{adj} - \rho \hat{\mu}_F^{adj}}{\gamma \sigma^2 (1 - \rho^2)} > 0
\]

Scenario 1: \( \hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \rho > 0, \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj}}{\hat{\mu}_U^{adj}} \)

Scenario 2: \( \hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \rho = 0 \)

Scenario 3: \( \hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} \geq 0, \rho < 0 \)

Scenario 4: \( \hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} < 0, \rho < 0, \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj}}{\hat{\mu}_U^{adj}} \)
Case 2, Participation in one asset

\[
\pi_F = \frac{\hat{\mu}_F^{adj}}{\gamma \sigma^2} > 0
\]
\[
\pi_U = 0
\]

Scenario 1: \(\hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \rho > 0, \frac{\hat{\mu}_F^{adj}}{\hat{\mu}_U^{adj}} \geq \frac{1}{\rho}\)
Scenario 2: \(\hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} \leq 0, \rho \geq 0\)
Scenario 3: \(\hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} < 0, \rho < 0, \frac{\hat{\mu}_F^{adj}}{\hat{\mu}_U^{adj}} \geq \frac{1}{\rho}\)

Case 3, Non-participation

\[
\pi_F = 0
\]
\[
\pi_U = 0
\]

Scenario 1: \(\hat{\mu}_F^{adj} = 0, \hat{\mu}_U^{adj} \leq 0\)
Scenario 2: \(\hat{\mu}_F^{adj} < 0, \hat{\mu}_U^{adj} < 0\)

In all three cases, when correlation decreases, familiar and unfamiliar assets are less correlated and diversification benefit increases. Thus, \(1/\rho\) represents diversification benefit. As \(\rho\) decreases, \(1/\rho\) increases, representing an increase in diversification benefit.

\[
1/\rho \equiv \text{diversification benefit}
\]

The difference between ambiguities is represented by \(\hat{\mu}_F^{adj} / \hat{\mu}_U^{adj}\). When the difference between ambiguities between familiar asset \(\theta_F\) and unfamiliar asset \(\theta_U\) is larger, \(\hat{\mu}_F^{adj} / \hat{\mu}_U^{adj}\) increases.

When the difference between ambiguities is larger, risk-adjusted return of unfamiliar asset decreases relative to that of familiar asset. This implies that diversification by holding unfamiliar asset is more costly; an investor has to bear worse risk-adjusted return of unfamiliar asset relative to that of familiar asset. Thus, difference between ambiguities represents
diversification cost. When the difference between ambiguities is higher, diversification cost is higher.

\[ \hat{\mu}_F^{adj} / \hat{\mu}_U^{adj} \equiv \text{the difference between ambiguities} \equiv \text{diversification cost} \]

**Case 1**

Participation in both assets happens in several scenarios.

Scenario 1, an investor always holds familiar asset because it has either equal or higher positive excess return than that of unfamiliar. In contrast, holding unfamiliar asset that has a positive correlation with familiar asset is risky. This is because when risk-adjusted return of familiar asset decreases, risk-adjusted return of unfamiliar asset also decreases; both assets move in the same direction. Then the investor compares between diversification benefit and diversification cost. Therefore, to hold unfamiliar asset, diversification benefit has to be greater than diversification cost, meaning that the difference between ambiguities is not large enough to cause non-holding of unfamiliar asset.

In Scenario 2, since both assets have positive excess return and have no correlation, the investor has no risk in holding both assets. Thus he receives high diversification benefit. As long as ambiguities are not sufficiently high to cause negative excess return, diversification benefit is always greater than diversification cost (the difference between ambiguities). Thus, both assets are held.

In Scenario 3, the risk-averse investor always prefers negative correlation because of hedging benefit; when risk-adjusted return of familiar asset decreases, risk-adjusted return of unfamiliar asset increases. This reduces the loss of investment in familiar asset when the loss
happens. Thus, the investor is willing to pay for this hedging benefit. There are many examples in the real world such as insurance and option. In this scenario, he does not pay for the hedging benefit but gains from it by having positive excess return of unfamiliar asset. Even when the excess return is zero, he still has hedging benefit for free. Therefore, he always holds both assets.

In Scenario 4, ambiguity of unfamiliar asset is sufficiently high which causes negative excess return. The investor has hedging benefit from negative correlation but he has to pay for it by having negative excess return of unfamiliar asset. Therefore, he only holds both assets when diversification benefit from hedging is greater than the diversification cost of holding unfamiliar asset. This implies that the difference between ambiguities is not large enough to cause non-holding of unfamiliar asset. This scenario is similar to an insurance. Unfamiliar asset acts like an insurance and its negative excess return is like a price of an insurance. He holds unfamiliar asset as an insurance only when the price of insurance is not too high and risk-adjusted return of familiar is high enough for him to benefit from having an insurance.

Looking at the weight equation, when ambiguity of an asset is higher, the investor reduces its weight. When return ambiguity of familiar asset increases, $\hat{\mu}_F^{adj}$ decreases, resulting in lower $\pi_F$. Similarly for unfamiliar asset, when return ambiguity of unfamiliar asset increases, $\hat{\mu}_U^{adj}$ decreases, resulting in lower $\pi_U$.

Weight of familiar asset also depends on ambiguity of unfamiliar asset. When ambiguity of unfamiliar asset increases, weight of familiar asset increases; Higher $\theta_U$ results in lower $\hat{\mu}_U^{adj}$ and thus increases $\pi_F$. Similarly, higher ambiguity of familiar asset increases the weight of unfamiliar asset. Higher $\theta_F$ results in lower $\hat{\mu}_F^{adj}$ and thus increases $\pi_U$. 

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When the difference between ambiguity of familiar and unfamiliar asset increases, 
\( \hat{\mu}_F^{adj} \) is relatively higher than \( \hat{\mu}_U^{adj} \) which increases \( \hat{\mu}_F^{adj} - \rho \hat{\mu}_U^{adj} \) and decreases \( \hat{\mu}_U^{adj} - \rho \hat{\mu}_F^{adj} \). As a result, \( \pi_F \) increases and \( \pi_U \) decreases. Intuitively, when the difference between ambiguities is higher, familiar asset is increasingly more attractive than unfamiliar asset. Thus, the investor has a bias toward familiar asset.

When ambiguities of assets are equal, equal weights are observed. As \( \hat{\mu}_U^{adj} = \hat{\mu}_F^{adj} \), the optimal weights are \( \pi_F = \pi_U = \hat{\mu}^{adj} / \gamma \sigma^2 (1 + \rho) \). Intuitively, both assets are equally attractive to the investor as they have equal risk-adjusted return.

**Case 2**

Participation in only familiar asset happens in several scenarios.

Case 2 Scenario 1 is the opposite of Case 1 Scenario 1. The investor has a risk when unfamiliar asset has a positive correlation. Then, he compares the cost and benefit of holding unfamiliar asset. When diversification cost is greater than diversification benefit, he ignores diversification benefit and holds only familiar asset. This implies that the difference between ambiguities is sufficiently large, causing non-holding of unfamiliar asset.

Case 2 Scenario 2 is the opposite of Case 1 Scenario 2 and Case 1 Scenario 3. Ambiguity of unfamiliar asset is sufficiently high which causes negative or zero excess return. In addition, it has a risk as it has a positive correlation with familiar asset. Thus the investor has no benefit in holding unfamiliar asset; he holds only familiar asset.

Case 2 Scenario 3 is the opposite of Case 1 Scenario 4. Ambiguity of unfamiliar asset is sufficiently high which causes negative excess return. The investor gains hedging benefit from
negative correlation but has to hold negative risk-adjusted return of unfamiliar asset. Thus, he holds only familiar asset when cost of diversification by holding unfamiliar asset is greater than hedging benefit. This implies that the difference between ambiguities is sufficiently large, causing non-holding of unfamiliar asset. This Scenario is similar to an insurance where the price of insurance is too high and risk-adjusted return of familiar is not high enough for him to benefit from having the insurance.

Looking at the weight equation, when ambiguity of familiar asset increases, its risk-adjusted return decreases and thus the investor puts less weight. Since he does not participate in unfamiliar asset, the weight of familiar asset does not depend on ambiguity of unfamiliar asset.

**Case 3**

Non-participation happens in two scenarios.

In Scenario 1 when excess return of unfamiliar asset is negative or zero and excess return of familiar asset is zero, the investor is better off by holding only risk-free asset. Although when excess return is zero, meaning it has the same return as risk-free asset, investing in risk-free asset is better as it has no risk. Thus, non-participation is observed.

In Scenario 2, when both excess return are negative, holding risk-free asset clearly gives a higher return with no risk. Thus the investor holds only risk-free asset and does not participate in any risky asset.

Excess returns could be negative from two situations. First, they are negative by themselves. Second, they are previously positive but their ambiguities are sufficiently high which make them negative. Thus, in the second situation, sufficiently high ambiguities of both assets cause non-participation.
Model 3

Case 1, Participation in both assets

\[
\begin{align*}
\pi_F &= \frac{\hat{\mu}(\sigma_{U}^{adj} - \rho\sigma_{F}^{adj})}{\gamma\sigma_{F}^{2adj}\sigma_{U}^{adj}(1 - \rho^2)} > 0 \\
\pi_U &= \frac{\hat{\mu}(\sigma_{F}^{adj} - \rho\sigma_{U}^{adj})}{\gamma\sigma_{U}^{2adj}\sigma_{F}^{adj}(1 - \rho^2)} > 0
\end{align*}
\]

Scenario 1: \( \hat{\mu} > 0, \rho > 0, \frac{1}{\rho} > \frac{\sigma_{U}^{adj}}{\sigma_{F}^{adj}} \)

Scenario 2: \( \hat{\mu} > 0, \rho \leq 0 \)

Case 2, Participation in one asset

\[
\begin{align*}
\pi_F &= \frac{\mu}{\gamma\sigma_{F}^{2adj}} > 0 \\
\pi_U &= 0
\end{align*}
\]

Scenario: \( \hat{\mu} > 0, \rho > 0, \frac{\sigma_{U}^{adj}}{\sigma_{F}^{adj}} \geq \frac{1}{\rho} \)

Case 3, Non-participation

\[
\begin{align*}
\pi_F &= 0 \\
\pi_U &= 0
\end{align*}
\]

Scenario: \( \hat{\mu} \leq 0 \)

Similar to Model 2, \( 1/\rho \) represents diversification benefit. The difference between ambiguities is represented by \( \frac{\sigma_{U}^{adj}}{\sigma_{F}^{adj}} \). When the difference between ambiguities is larger, \( \frac{\sigma_{U}^{adj}}{\sigma_{F}^{adj}} \) increases.

Similar to Model 2, the difference between ambiguities represents the cost of diversification. This is because when the difference between ambiguities is larger, an investor has to bear worse risk-adjusted return of unfamiliar asset relative to that of familiar asset.

Therefore, when the difference between ambiguities is higher, diversification cost is higher.

\[
\frac{\sigma_{U}^{adj}}{\sigma_{F}^{adj}} \equiv \text{the difference between ambiguities} \equiv \text{diversification cost}
\]
Case 1

Scenario 1: Holding unfamiliar asset is risky because of a positive correlation with familiar asset. Then the investor compares between diversification benefit and diversification cost. Therefore, he holds unfamiliar asset when diversification benefit of holding unfamiliar asset is greater than diversification cost, meaning that the difference between ambiguities is not large enough to cause non-holding of unfamiliar asset.

Scenario 2: The investor has hedging benefit from a negative correlation and high diversification benefit from zero correlation. In addition, he benefits from positive excess returns. Thus he always holds both assets.

Looking at the weight equation, variance ambiguity reduces the weight of the ambiguity-asset. For familiar asset, when ambiguity $\theta_F$ increases, $\sigma_F^{adj}$ increases, resulting in lower $\pi_F$. Similar for unfamiliar asset, when ambiguity $\theta_U$ increases, $\sigma_U^{adj}$ increases, resulting in lower $\pi_U$.

Weight of familiar asset depends on ambiguity of unfamiliar asset; when variance ambiguity of unfamiliar asset increases, weight of familiar asset increases. To see this, $\pi_F$ and $\pi_U$ are rearranged as

$$
\pi_F = \frac{\hat{\mu}}{\gamma \sigma_F^{adj} (1 - \rho^2)} \left( \frac{1}{\sigma_F^{adj} - \rho \sigma_U^{adj}} \right)
$$

$$
\pi_U = \frac{\hat{\mu}}{\gamma \sigma_U^{adj} (1 - \rho^2)} \left( \frac{1}{\sigma_U^{adj} - \rho \sigma_F^{adj}} \right)
$$
As $\theta_U$ increases, $\sigma_U^{adj}$ increases, resulting in higher $\pi_F$. Similarly, when variance ambiguity of familiar asset increases, weight of unfamiliar asset increases. As $\theta_F$ increases, $\sigma_F^{adj}$ increases, resulting in higher $\pi_U$. Intuitively, when ambiguity of unfamiliar asset increases, familiar asset becomes more attractive because of relatively higher risk-adjusted return.

When ambiguities of assets are equal, equal weights are observed. As $\sigma_U^{adj} = \sigma_F^{adj}$, the optimal weights are $\pi_F = \pi_U = \hat{\mu}/\gamma \sigma^{2adj} (1 + \rho)$. Intuitively, both assets are equally attractive to the investor as they have equal risk-adjusted return.

**Case 2**

Case 2 Scenario is the opposite of Case 1 Scenario 1. Positive correlation between assets is risky. The investor will compare diversification benefit and diversification cost. He holds only familiar asset when diversification cost is greater than diversification benefit, meaning that the difference between ambiguities is sufficiently large, causing non-holding of unfamiliar asset.

Looking at the weight equation, when ambiguity of familiar asset increases, its risk-adjusted return decreases and thus the investor puts less weight. Since he does not participate in unfamiliar asset, the weight of familiar asset does not depend on ambiguity of unfamiliar asset.

**Case 3**

When excess returns are negative or zero, holding risk-free asset gives a higher return or equal return with no risk. Thus, non-participation is observed. Variance ambiguity never causes non-participation since it could not cause excess return to be negative. Intuitively, no matter how high a variance of an asset is, its risk-adjusted return is still positive.
Model 4

Case 1, Participation in both assets

\[
\pi_F = \frac{\mu_F^{adj} \sigma_U^{adj} - \hat{\mu}_U^{adj} \rho \sigma_F^{adj}}{\gamma \sigma_F^{2adj} \sigma_U^{adj} (1 - \rho^2)} > 0
\]

\[
\pi_U = \frac{\mu_U^{adj} \sigma_F^{adj} - \hat{\mu}_F^{adj} \rho \sigma_U^{adj}}{\gamma \sigma_U^{2adj} \sigma_F^{adj} (1 - \rho^2)} > 0
\]

Scenario 1: \( \mu_F^{adj} \geq \mu_U^{adj} > 0, \rho > 0, 1 / \rho > \frac{\mu_F^{adj}}{\mu_U^{adj}} \sqrt{\frac{\sigma_U^{adj}}{\sigma_F^{adj}}} \)

Scenario 2: \( \mu_F^{adj} \geq \mu_U^{adj} > 0, \rho = 0 \)

Scenario 3: \( \mu_F^{adj} > 0, \mu_U^{adj} \geq 0, \rho < 0 \)

Scenario 4: \( \mu_F^{adj} > 0, \mu_U^{adj} < 0, \rho < 0, 1 / \rho > \frac{\mu_F^{adj}}{\mu_U^{adj}} \sqrt{\frac{\sigma_U^{adj}}{\sigma_F^{adj}}} \)

Case 2, Participation in one asset

\[
\pi_F = \frac{\mu_F^{adj}}{\gamma \sigma_F^{2adj}} > 0
\]

\[
\pi_U = 0
\]

Scenario 1: \( \mu_F^{adj} \geq \mu_U^{adj} > 0, \rho > 0, 1 / \rho > \frac{\mu_F^{adj}}{\mu_U^{adj}} \sqrt{\frac{\sigma_U^{adj}}{\sigma_F^{adj}}} \)

Scenario 2: \( \mu_F^{adj} > 0, \mu_U^{adj} \leq 0, \rho \geq 0 \)

Scenario 3: \( \mu_F^{adj} > 0, \mu_U^{adj} < 0, \rho < 0, 1 / \rho > \frac{\mu_F^{adj}}{\mu_U^{adj}} \sqrt{\frac{\sigma_U^{adj}}{\sigma_F^{adj}}} \)

Case 3, Non-participation

\[
\pi_F = 0
\]

\[
\pi_U = 0
\]

Scenario 1: \( \mu_F^{adj} = 0, \mu_U^{adj} \leq 0 \)

Scenario 2: \( \mu_F^{adj} < 0, \mu_U^{adj} < 0 \)

Similarly, \( 1 / \rho \) represents diversification benefit. The difference between ambiguities is represented by \( \mu_F^{adj} \sigma_U^{adj} / \mu_U^{adj} \sigma_F^{adj} \). When the difference between ambiguities between familiar and unfamiliar is larger, \( \mu_F^{adj} \sigma_U^{adj} / \mu_U^{adj} \sigma_F^{adj} \) increases.
Similar to Model 2, the difference between ambiguities represents the cost of diversification. This is because when the difference between ambiguities is larger, an investor has to bear worse risk-adjusted return of unfamiliar asset relative to that of familiar asset. Therefore, when the difference between ambiguities is higher, diversification cost is higher.

\[
\hat{\mu}_F^{adj} \sigma_U^{adj} / \hat{\mu}_U^{adj} \sigma_F^{adj} \equiv \text{the difference between ambiguities} \equiv \text{diversification cost}
\]

Case 1

In Scenario 1, Investor always holds familiar asset because of higher positive excess return than that of unfamiliar asset. Meanwhile, holding unfamiliar asset is risky because of positive correlation. Then, he will compare the cost and benefit of diversification. He holds unfamiliar asset when diversification benefit is greater than diversification cost, meaning that the difference between ambiguities is not large enough to cause non-holding of unfamiliar asset.

In Scenario 2, since both assets have positive excess returns and have no correlation, the investor has no risk in holding both assets and receives high diversification benefit. As long as ambiguities of both assets are not high enough to cause negative excess returns, diversification benefit is always greater than diversification cost. As a result, he holds both assets.

In Scenario 3, the risk-averse investor always prefers negative correlation because of hedging benefit as it reduces the loss of investment in familiar asset. Thus, he is willing to pay for this hedging benefit. In this scenario, he does not pay for the hedging benefit but gains from it by having positive excess return of unfamiliar asset. Even when the excess return is zero, he still has hedging benefit for free. Therefore, he always holds both assets.
In Scenario 4, the investor has hedging benefit from negative correlation but he has to pay for it by having negative excess return of unfamiliar asset which is caused by sufficiently high ambiguity. Therefore, he only holds both assets when diversification benefit from hedging is greater than the diversification cost of holding unfamiliar asset. This means that the difference between ambiguities is not large enough to cause non-holding of unfamiliar asset. In this scenario, unfamiliar asset acts like an insurance and its negative excess return is like a price of an insurance.

Looking at the weight equation, higher return and variance ambiguities of an asset lowers its weight. For familiar asset, when ambiguity of familiar asset increases, $\hat{\mu}_F^{adj}$ decreases and $\sigma_F^{adj}$ increases, resulting in lower $\pi_F$. Similarly for unfamiliar asset, when return ambiguity of unfamiliar asset increases, $\hat{\mu}_U^{adj}$ decreases and $\sigma_U^{adj}$ increases, resulting in lower $\pi_U$.

Weight of familiar asset also depends on ambiguity of unfamiliar asset. When ambiguity of unfamiliar asset increases, weight of familiar asset increases; Higher $\theta_U$ causes lower $\hat{\mu}_U^{adj}$ and higher $\sigma_U^{adj}$, resulting in higher $\pi_F$. Similarly, higher ambiguity of familiar asset increases the weight of unfamiliar asset. Higher $\theta_F$ causes lower $\hat{\mu}_F^{adj}$ and higher $\sigma_F^{adj}$, resulting in higher $\pi_U$. Intuitively, when ambiguity of unfamiliar asset increases, familiar asset becomes more attractive because of relatively higher risk-adjusted return.

When ambiguities of assets are equal, equal weights are observed. As $\hat{\mu}_U^{adj} = \hat{\mu}_F^{adj}$ and $\sigma_U^{adj} = \sigma_F^{adj}$, the optimal weights are $\pi_F = \pi_U = \hat{\mu}^{adj} / \gamma \sigma^{2 adj} (1 + \rho)$. Intuitively, both assets are equally attractive to the investor as they have equal risk-adjusted return.
Case 2

Case 2 Scenario 1 is the opposite of Case 1 Scenario 1. The investor has a risk when unfamiliar asset has a positive correlation. He compares the cost and benefit of holding unfamiliar asset. When diversification cost is greater than diversification benefit, he ignores diversification benefit and holds only familiar asset, meaning that the difference between ambiguities is sufficiently large, causing non-holding of unfamiliar asset.

Case 2 Scenario 2 is the opposite of Case 1 Scenario 2 and Case 1 Scenario 3. Ambiguity of unfamiliar asset is sufficiently high causing its excess return to be negative or zero. Moreover, unfamiliar asset has a positive correlation with familiar asset. Thus the investor has no benefit in holding unfamiliar asset; he holds only familiar asset.

Case 2 Scenario 3 is the opposite of Case 1 Scenario 4. An investor gains hedging benefit from negative correlation but has to hold negative risk-adjusted return of unfamiliar asset which is caused by its high ambiguity. In this scenario, he holds only familiar asset when cost of diversification by holding unfamiliar asset is greater than diversification benefit, meaning that the difference between ambiguities is sufficiently large, causing non-holding of unfamiliar asset. This scenario is similar to an insurance when the price of insurance is too high and risk-adjusted return of familiar is not high enough for him to benefit from having the insurance.

Looking at the weight equation, when ambiguity of familiar asset increases, its risk-adjusted return decreases and thus the investor puts less weight. Since he does not participate in unfamiliar asset, the weight of familiar asset does not depend on ambiguity of unfamiliar asset.
Case 3

In Scenario 1 when excess return of unfamiliar asset is negative or zero and excess return of familiar asset is zero, the investor is better off by holding only risk-free asset. Although when excess return is zero, meaning it has the same return as risk-free asset, investing in risk-free asset has no risk. Thus, non-participation is observed.

In Scenario 2, when both excess returns are negative, holding risk-free asset clearly gives a higher return with no risk. Thus the investor holds only risk-free asset and does not participate in any risky asset.

Excess returns could be negative from two situations. First, they are negative by themselves. Second, they are previously positive but their ambiguities are sufficiently high which make them negative. Thus, in the second situation, sufficiently high ambiguities cause non-participation.

Comparison among models

The mean-variance investor prefers high excess return and low variance of an asset. In Model 2, when return ambiguity of the asset is higher, its excess return decreases and thus risk-adjusted return decreases. He finds the asset less attractive and thus decreases the holding of the asset. Similarly, in Model 3, when variance ambiguity of the asset increases, its variance is higher. Its risk-adjusted return is lower and the asset becomes less attractive and thus the holding of the asset is reduced. In Model 4, higher return and variance ambiguities together result in lower excess return and higher variance. The effect of ambiguity in this model is a combination of Model 2’s and Model 3’s. This is the opposite of what the mean-variance investor prefers and thus he finds the asset more unattractive than in Model 2 and Model 3. Therefore, he further
decreases the holding of unfamiliar asset. Given equal ambiguities among the four models, weight in Model 4 is the lowest because the effect of ambiguity is the greatest.

For all models, the difference between ambiguities implies higher diversification cost of holding unfamiliar asset. When the difference between ambiguities is higher, risk-adjusted return of familiar asset is relatively more attractive than that of unfamiliar asset. Thus, the investor reduces the holding of unfamiliar asset. This explains why ambiguities cause poor diversification.

The effect of ambiguities is seen in three cases. In Case 1, although the investor holds both assets, higher ambiguity of unfamiliar asset causes a bias investment toward familiar asset. In this case, he follows both Keynes’s advice by focusing on familiar asset and Markowitz’s advice by spreading his capital among other assets.

In Case 2, ambiguity causes extreme participation in familiar asset and no participation in unfamiliar asset by two ways. First, the difference between ambiguities of both assets is sufficiently large. Second, ambiguity of unfamiliar asset itself is sufficiently high, causing negative excess return. In this case, the investor holds assets by Keynesian’s suggestion to the extreme by focusing only on the asset he knows about and ignores diversification benefit by Markowitz.

For Case 3, sufficiently high return ambiguities of both assets cause their excess return to be negative. This happens only in Model 2 and Model 4 because they have return ambiguity. In contrast, Model 3 does not have this situation because variance ambiguity never causes negative excess return. Thus, non-participation is not caused by variance ambiguity.
As in the real world non-participation in many cases is caused by uncertainty, Model 3 is not realistic. Moreover, return is more difficult to estimate than variance. Thus Model 3 which incorporates only variance ambiguity and ignores return ambiguity is inaccurate. Although Model 2 and Model 4 show similar cases, it is not true to assume that an investor has no ambiguity in variance. Thus Model 4 is the most realistic.

4.2 Relative weight

To see the effect of ambiguity on relative investment of the risky assets, relative weights of familiar and unfamiliar assets, \( w_F \) and \( w_U \), are used. Relative weight is calculated as

\[
 w_F = \frac{\pi_F}{\pi_F + \pi_U} \\
 w_U = \frac{\pi_U}{\pi_F + \pi_U}
\]

Intuitively, \( w_F \) shows a percentage investment in familiar asset relative to the total investment in all risky assets. Similarly, \( w_U \) shows a percentage investment in unfamiliar asset relative to the total investment in all risky assets. When \( \pi_F \) increases, \( w_F \) increases and \( w_U \) decreases. When \( \pi_U \) increases, \( w_U \) increases and \( w_F \) decreases. Relative weight of unfamiliar asset could also be obtained by \( 1 - w_F \) since the sum of weight is 100%. Relative weights in each model are as follow:
**Model 1**

Case 1, Participation in both assets

\[ w_F = 50\% \]
\[ w_U = 50\% \]

Case 2, Non-participation

\[ w_F = 0 \]
\[ w_U = 0 \]

In Case 1 relative weights are divided equally as two assets are equal. In Case 2 non-participation implies zero weights of both assets.

**Model 2**

Case 1, Participation in both assets

\[
\begin{align*}
W_F &= \frac{\hat{\mu}_F^{adj} - \rho \hat{\mu}_U^{adj}}{(1 - \rho)(\hat{\mu}_F^{adj} + \hat{\mu}_U^{adj})} > 0 \\
W_U &= \frac{\hat{\mu}_U^{adj} - \rho \hat{\mu}_F^{adj}}{(1 - \rho)(\hat{\mu}_F^{adj} + \hat{\mu}_U^{adj})} > 0
\end{align*}
\]

Case 2, Participation in one asset

\[ w_F = 100\% \]
\[ w_U = 0 \]

Case 3, Non-participation

\[ w_F = 0 \]
\[ w_U = 0 \]

In Case 1 both relative weights are positive and relative weight of familiar asset is higher than that of unfamiliar asset when there is a difference between ambiguities. In Case 2 relative weight of familiar asset is 100% since there is no participation in unfamiliar asset. In Case 3, non-participation implies zero weights of both assets.
Model 3

Case 1, Participation in both assets

\[ w_F = \frac{\sigma_u^{adj} - \rho \sigma_F^{adj}}{(1 - \rho)(\sigma_u^{adj} + \sigma_F^{adj})} > 0 \]

\[ w_F = \frac{\sigma_F^{adj} - \rho \sigma_u^{adj}}{(1 - \rho)(\sigma_u^{adj} + \sigma_F^{adj})} > 0 \]

Case 2, Participation in one asset

\[ w_F = 100\% \]
\[ w_U = 0 \]

Case 3, Non-participation

\[ w_F = 0 \]
\[ w_U = 0 \]

In Case 1 both relative weights are positive and relative weight of familiar asset is higher than that of unfamiliar asset when there is a difference between ambiguities. In Case 2 relative weight of familiar asset is 100% since there is no participation in unfamiliar asset. In Case 3, non-participation implies zero weights of both assets.

Model 4

Case 1, Participation in both assets

\[ w_F = \frac{\hat{\mu}_F^{adj} \sigma_u^{adj} - \hat{\mu}_U^{adj} \rho \sigma_F^{adj}}{(1 - \rho)(\hat{\mu}_F^{adj} \sigma_u^{adj} + \hat{\mu}_U^{adj} \sigma_F^{adj})} > 0 \]

\[ w_U = \frac{\hat{\mu}_U^{adj} \sigma_F^{adj} - \hat{\mu}_F^{adj} \rho \sigma_U^{adj}}{(1 - \rho)(\hat{\mu}_F^{adj} \sigma_u^{adj} + \hat{\mu}_U^{adj} \sigma_F^{adj})} > 0 \]

Case 2, Participation in one asset

\[ w_F = 100\% \]
\[ w_U = 0 \]
Case 3, Non-participation

\[ w_F = 0 \]
\[ w_U = 0 \]

In Case 1 both relative weights are positive and relative weight of familiar asset is higher than that of unfamiliar asset when there is a difference between ambiguities. In Case 2 relative weight of familiar asset is 100% since there is no participation in unfamiliar asset. In Case 3, non-participation implies zero weights of both assets.

5. Data and a numerical example

Similar to Boyle et al (2010), we test the effect of ambiguity in all models with a numerical example by assuming two equal risky assets having the same excess return \( \hat{\mu} = 0.05 \), variance \( \hat{\sigma}^2 = 0.07 \), and have a correlation between them \( \rho = 0.5 \). The number of observation \( T = 70 \) and risk aversion \( \gamma = 2 \) are used.

The effect of ambiguities is reported in Tables in Appendix. All Tables show the effect of ambiguity on relative weight of familiar asset \( w_F \) by varying ambiguities of both assets from 0 to 9 with an increment of 0.1. Ambiguity of familiar asset, \( \theta_F \) is the leftmost column and ambiguity of unfamiliar asset, \( \theta_U \) is the top row. NP stands for “no participation” in any asset. Relative weight of unfamiliar asset is calculated by \( 1 - w_F \).

Model 1

The classical model gives two equal relative weights for both assets \( w_F = w_U = 50\% \) because both assets have equal positive excess returns.
Case 1 Scenario 1 is observed. When diversification benefit is higher than the difference between ambiguities (diversification cost), the model shows positive weights in both assets with a bias toward familiar asset in various combinations of ambiguities in Table 1. When return ambiguities of two assets are equal, equal relative weights are observed. For example, when $\theta_F = 0.3$ and $\theta_U = 0.3$, the investor holds each asset $w_F = w_U = 50\%$.

Case 2 Scenario 1 is observed. When the difference between ambiguities (diversification cost) is larger than diversification benefit, risk-adjusted return of familiar asset is sufficiently larger than that of unfamiliar asset and thus the investor ignores diversification benefit and thus does not hold any unfamiliar asset. For example, when $\theta_U = 0.6$ and $\theta_F = 0$, difference between ambiguities $\hat{\mu}_F^{adj}/\hat{\mu}_U^{adj} = 2.14$ is greater than diversification benefit $1/\rho = 2$. Thus the investor does not invest in unfamiliar asset at all and holds $w_F = 100\%$.

Case 2 Scenario 2 is observed. When ambiguity of unfamiliar asset is sufficiently high making its excess return negative while ambiguity of familiar asset is not high enough to make its excess return negative, the investor participates in only familiar asset. This happens when $\theta_U = 0.9$ ($\hat{\mu}_U^{adj} = -0.002$) and $\theta_F < 0.9$ ($\hat{\mu}_F^{adj} > 0$).

Case 3 Scenario 2 is observed. When ambiguities of both assets are sufficiently high, risk-adjusted return of both assets are negative. Thus the investor does not hold risky assets and non-participation is observed. This happens when $\theta_U = 0.9$ and $\theta_F = 0.9$. Excess returns of both assets are negative $\hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = -0.002$ and thus $w_F = w_U = 0$. 

Model 2
The effect of return ambiguity lowers participation in the ambiguity-asset. For instance, when return ambiguity of unfamiliar asset increases, the investor puts more relative weight in familiar asset and less relative weight in unfamiliar asset. For example, given $\theta_F = 0$, when $\theta_U$ increases from 0 to 0.5, the investor increases the holding of familiar asset $w_F$ from 50% to 91% and decreases the holding of unfamiliar asset $w_U$ from 50% to 9%.

**Model 3**

Only Case 1 Scenario 1 is observed in Table 2, implying in every combination of ambiguities, diversification benefit is always higher than the difference between ambiguities (diversification cost). To see why only Case 1 happens, let’s look at the highest difference between ambiguities. This happens when $\theta_F = 0$ and $\theta_U = 0.9$ in which the difference between ambiguities $\sigma_U^{adj}/\sigma_F^{adj} = 1.34$ is less than diversification benefit $1/\rho = 2$. Thus, positive relative weights are observed. As Case 2 requires that the difference between ambiguities is greater than diversification benefit, thus Case 2 does not happen. Case 3 requires that excess return are negative. Although higher variance ambiguity decreases risk-adjusted return, it never makes excess return negative. Thus Case 3 does not happen.

Similar to return ambiguity, variance ambiguity of an asset causes lower participation in that asset. For instance, when variance ambiguity of unfamiliar asset increases, the investor puts more relative weight in familiar asset and less relative weight in unfamiliar asset. For example, given ambiguity $\theta_F = 0$, when $\theta_U$ increases from 0 to 0.9, the investor increases the relative holding of the familiar asset from $w_F = 50\%$ to $w_F = 64\%$ and decreases the relative holding of unfamiliar asset form $w_U = 50\%$ to $w_U = 36\%$. 

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Model 4

Since Model 4 has both return and variance ambiguities, we explore the case when investor has equal ambiguities in return and variance. For instance, ambiguity $\theta = 0.4$ means return ambiguity is 0.4 and variance ambiguity is 0.4.

Case 1 Scenario 1 is observed in various combinations of ambiguities in Table 3. This happens when diversification benefit is higher than the difference between ambiguities (diversification cost). When ambiguities of two assets are equal, both assets have the same excess return and variance. Thus the investor puts equal relative weights. For example, when $\theta_F = 0.4$ and $\theta_U = 0.4$, the investor holds each asset $w_F = w_U = 50\%$.

Case 2 Scenario 1 is observed. When the difference between ambiguities (diversification cost) is higher than diversification benefit, the investor does not participate in unfamiliar asset. For example, when $\theta_U = 0.7$, the investor who has $\theta_F = 0.3$ puts $w_U = 0\%$ and $w_F = 100\%$. This is because the difference between ambiguities $\frac{\hat{\mu}_F^{adj} - \hat{\mu}_U^{adj}}{\sigma_U^{adj} / \sigma_F^{adj}} = 2.46$ is greater than diversification benefit $1/\rho = 2$.

Case 2 Scenario 2 is observed. When ambiguity of unfamiliar asset is sufficiently high making its excess return negative while ambiguity of familiar asset is not high enough to make its excess return negative, the investor participates in only familiar asset. This happens when $\theta_U = 0.9$ ($\mu_U^{adj} = -0.002$) and $\theta_F < 0.9$ ($\mu_F^{adj} > 0$).

Case 3 Scenario 2 is observed. When the difference between ambiguities for both assets is sufficiently high, excess returns are negative. Thus, the investor does not invest in any risky
asset and non-participation is observed. For example, when $\theta_U = 0.9$ and $\theta_F = 0.9$, excess returns of both assets are negative $\hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = -0.002$. Thus, $w_F = w_U = 0$.

The effect of ambiguity is similar to Model 2 and Model 3. The investor who has both return and variance ambiguities together decreases his participation in the ambiguity-asset when its ambiguity increases. For instance when ambiguity of unfamiliar asset increases, the investor holds relatively more familiar asset and relatively less on unfamiliar asset. For example, given $\theta_F = 0$, when $\theta_U$ increases from 0 to 0.5, the investor increases the holding of familiar asset $w_F$ from 50% to 95% and decreases that of unfamiliar asset $w_U$ from 50% to 5%.

Model 4 Special Case

In this case, we assume return ambiguity is higher than variance ambiguity because Garlappi, Uppal and Wang (2007) claims that return is notoriously more difficult to estimate than variance. In our setting, return ambiguity is double variance ambiguity. For example, when return ambiguity is 0.5, variance ambiguity is 0.25. Since Model 4 normal case and special case are quite similar, we focus on the difference between their results.

Comparing to a normal case, the investor puts less relative weight on familiar asset because of lower variance ambiguity in Special case. For example, in Table 4 when return and variance ambiguities of unfamiliar asset are 0.4, 0.2 and those of familiar asset are 0.1, 0.05 consecutively, the investor puts $w_F = 75\%$. Meanwhile in the normal case in Table 2, when return and variance ambiguities of unfamiliar asset are 0.4 and those of familiar asset are 0.1, the investor puts $w_F = 77\%$. Due to greater effect of ambiguities in normal case, participation in only one asset is observed more frequently than in the special case.
6. Empirical evidence

Bias toward familiar asset in Case 1 explains empirical observation that employees put relatively higher amount of their wealth in the stocks of their own-companies than in other companies’ stocks. This is because employees are more familiar with their own-company and thus have less ambiguity. This consistently supports the empirical findings by Benartzi (2001), Mitchelle and Utkus (2002) and Meulbroek (2005) who find that employees invest discretionarily in their own-company’s stocks.

In addition, bias toward familiar asset explains home-bias; the investors hold relatively higher portion of home-country stocks than international stocks. This is because they are more familiar with stocks in their own countries than foreign stocks.

Participation in only familiar asset in Case 2 explains empirical observation of extreme home-bias; a large number of investors hold only their home-country stocks. This is because ambiguities of international assets are sufficiently high, supporting the finding of Cooper and Kaplanis (1994) who observe that investors hold home equity instead of diversifying across international equities.

Moreover, non-participation in Case 3 is supported by empirical studies of Vissing-Jorgensen (2003) and Campbell (2006) showing completely non-participation in the risky assets among investors who are sufficiently unfamiliar with all risky assets. Thus, they allocate their entire fund in the risk-free asset.
7. Conclusion

Our Model 4 is the most realistic as it has both return and variance ambiguities. Model 1 does not involve in ambiguities, thus it is not realistic and accurate. Variance ambiguity in Model 3 does not cause non-participation. In addition, it does not have return ambiguity. As uncertainty causes non-participation in the real world and return is much more difficult to estimate than variance, Model 3 is not realistic and accurate. Although Model 2 shows similar results as our model, it does not have variance ambiguity. It is wrong to assume that variance could be perfectly estimated. Thus, Model 2 is not realistic and accurate.

This paper shows that ambiguities is the reason why the mean variance investors in the real world have poor diversification by holding few assets and ignoring the diversification benefit in a large number of assets. This is because higher return and variance ambiguities imply lower risk-adjusted return of an asset. Thus holding of the asset is reduced. In addition, higher the difference between ambiguities implies higher diversification cost. Model 4 shows that return and variance ambiguities cause poor diversification in three cases. First, they cause a bias investment in familiar asset but unfamiliar asset is still held. Second, they cause participation in only familiar asset. Last, they cause non-participation in any risky asset.

The result of our model is consistent with empirical evidence, explaining why employees put more wealth in their own-companies’ stocks than those of other companies. In addition, the result explains poor diversification in international assets and a bias investment in home equities. It also explains extreme home-bias, investment in only home-equity. Moreover, it shows how uncertainty causes non-participation in the real world.
Therefore, we propose that classical model should be adjusted by including return and variance ambiguities. In fact, the classical model is a special case of our Model 4. We further propose that our model and its return and variance ambiguities should be used in constructing portfolio and in asset pricing. For future research, the case with N risky assets, the case with an infinite number of risky assets, the case with N risky assets with different asset classes and their derivation of the optimal portfolio weights are recommended to be explored.
Appendix

Table 1: Model 2 effect of ambiguity

The table shows relative weight of familiar asset, $w_F$ with varying ambiguities of familiar and unfamiliar asset. It shows ambiguity of familiar asset, $\theta_F$ in the leftmost column and ambiguity of unfamiliar asset, $\theta_U$ in the top row. NP stands for “no participation” in any asset. Relative weight of unfamiliar asset is $1 - w_F$.

<table>
<thead>
<tr>
<th>Ambiguity</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
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Table 2: Model 3 effect of ambiguity

The table shows relative weight of familiar asset, $w_F$ with varying ambiguities of familiar and unfamiliar asset. It shows ambiguity of familiar asset, $\theta_F$ in the leftmost column and ambiguity of unfamiliar asset, $\theta_U$ in the top row. NP stands for “no participation” in any asset. Relative weight of unfamiliar asset is $1 - w_F$.

<table>
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Table 3: Model 4 effect of ambiguity

The table shows relative weight of familiar asset, $w_F$ with varying ambiguities of familiar and unfamiliar asset. It shows ambiguity of familiar asset, $\theta_F$ in the leftmost column and ambiguity of unfamiliar asset, $\theta_U$ in the top row. NP stands for “no participation” in any asset. Relative weight of unfamiliar asset is $1 - w_F$.

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Table 4: Model 4 Special Case effect of ambiguity

The table shows relative weight of familiar asset, $w_F$ with varying ambiguities of familiar and unfamiliar asset. It shows ambiguity of familiar asset, $\theta_F$ in the leftmost column and ambiguity of unfamiliar asset, $\theta_U$ in the top row. NP stands for “no participation” in any asset. Relative weight of unfamiliar asset is $1 - w_F$.

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Derivation of the optimal weights

Model 4

We will begin with Model 4 because other models are special cases of Model 4. Model 2 is similar to Model 4 with equal volatilities. Model 3 is similar to Model 4 with equal excess return. Model 1 is similar to Model 4 with equal excess return and volatilities.

The optimization problem for two risky assets is

$$\max_{\pi_F, \pi_U} \pi_F \hat{\mu}_F^{adj} + \pi_U \hat{\mu}_U^{adj} - \frac{\gamma}{2} \left( \pi_F^2 \sigma_F^{adj} + \pi_U^2 \sigma_U^{adj} + 2 \pi_F \pi_U \rho \sigma_U^{adj} \sigma_F^{adj} \right)$$

Subject to

$$\pi_F \geq 0, \pi_U \geq 0$$

Thus,

$$L = \pi_F \hat{\mu}_F^{adj} + \pi_U \hat{\mu}_U^{adj} - \frac{\gamma}{2} \left( \pi_F^2 \sigma_F^{adj} + \pi_U^2 \sigma_U^{adj} + 2 \pi_F \pi_U \rho \sigma_U^{adj} \sigma_F^{adj} \right) + A \pi_F + B \pi_U$$

FOC

$$\hat{\mu}_F^{adj} - \gamma \pi_F \sigma_F^{adj} - \gamma \pi_U \rho \sigma_U^{adj} + A = 0 \quad (R1)$$

$$\hat{\mu}_U^{adj} - \gamma \pi_U \sigma_U^{adj} - \gamma \pi_F \rho \sigma_F^{adj} + B = 0 \quad (R2)$$

$$\pi_F \geq 0, \quad \pi_U \geq 0, \quad A \geq 0, \quad B \geq 0, \quad \pi_F A = 0, \quad \pi_U B = 0$$

There are several situations depending on $A$, $B$, $\pi_F$, $\pi_U$. We will first proves situations that do not exist.

Non-existing situations

Situation 1: $A = 0, B = 0, \pi_F = 0, \pi_U > 0$

Solving R1 and R2 gives

$$\pi_F = 0$$

$$\pi_U = \frac{\hat{\mu}_U^{adj} - \hat{\mu}_F^{adj}}{\gamma \sigma_U^{adj} (\sigma_U^{adj} - \rho \sigma_F^{adj})}$$

This happens when $\hat{\mu}_U^{adj} - \hat{\mu}_F^{adj} > 0, \sigma_U^{adj} - \rho \sigma_F^{adj} > 0$. Thus $\hat{\mu}_U^{adj} > \hat{\mu}_F^{adj}$. Since $\hat{\mu}_U^{adj} \leq \hat{\mu}_F^{adj}$, we have a contradiction.

This also happens when $\hat{\mu}_U^{adj} - \hat{\mu}_F^{adj} < 0, \sigma_U^{adj} - \rho \sigma_F^{adj} < 0$. Thus $\sigma_U^{adj} < \rho \sigma_F^{adj}$. Since $\sigma_U^{adj} > \rho \sigma_F^{adj}$, we have a contradiction.
Situation 2: $A > 0, B = 0, \pi_F = 0, \pi_U > 0$

Solving R1 and R2 gives

$$\pi_F = 0$$

$$\pi_U = \frac{\hat{\mu}_U^{adj} - \hat{\mu}_F^{adj} - A}{\gamma \sigma_U^{adj} (\sigma_U^{adj} - \rho \sigma_F^{adj})}$$

This happens when $\hat{\mu}_U^{adj} - \hat{\mu}_F^{adj} - A > 0$, $\sigma_U^{adj} - \rho \sigma_F^{adj} > 0$. Thus $\hat{\mu}_U^{adj} - \hat{\mu}_F^{adj} > A > 0$.

Since $\hat{\mu}_U^{adj} - \hat{\mu}_F^{adj} \leq 0$, we have a contradiction.

This also happens when $\hat{\mu}_U^{adj} - \hat{\mu}_F^{adj} - A < 0$, $\sigma_U^{adj} - \rho \sigma_F^{adj} < 0$. Thus $\sigma_U^{adj} < \rho \sigma_F^{adj}$. Since $\sigma_U^{adj} > \rho \sigma_F^{adj}$, we have a contradiction.

Situation 3: $A > 0, B = 0, \pi_F = 0, \pi_U = 0$

Solving R1 and R2 gives

$\hat{\mu}_F^{adj} + A = 0, \hat{\mu}_U^{adj} = 0$. Thus $\hat{\mu}_F^{adj} = -A < 0$. Since $\hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj}$, we have a contradiction.

Existing Situations

Situation 4: $A = 0, B > 0, \pi_F = 0, \pi_U = 0$

Solving R1 and R2 gives

$\hat{\mu}_U^{adj} + B = 0, \hat{\mu}_F^{adj} = 0$. Thus $\hat{\mu}_U^{adj} = -B < 0$. Thus we have condition $\hat{\mu}_F^{adj} = 0$, $\hat{\mu}_U^{adj} < 0$.

Situation 5: $A = 0, B = 0, \pi_F = 0, \pi_U = 0$

Solving R1 and R2 gives $\hat{\mu}_F^{adj} = 0$, $\hat{\mu}_U^{adj} = 0$. We have a condition.

Situation 6: $A > 0, B > 0, \pi_F = 0, \pi_U = 0$

Solving R1 and R2 gives

$\hat{\mu}_U^{adj} + B = 0, \hat{\mu}_F^{adj} + A = 0$. Thus $\hat{\mu}_U^{adj} = -B < 0, \hat{\mu}_F^{adj} = -A < 0$. Thus we have condition $\hat{\mu}_U^{adj} < 0, \hat{\mu}_F^{adj} < 0$. 

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Combining Situation 4, Situation 5, and Situation 6, we have

Optimal weights:

\[ \pi_F = 0 \]
\[ \pi_U = 0 \]

Scenario 1:

\[ \hat{\mu}_F^{adj} = 0, \hat{\mu}_U^{adj} \leq 0 \]

Scenario 2:

\[ \hat{\mu}_F^{adj} < 0, \hat{\mu}_U^{adj} < 0 \]

This proves Case 3 Model 4

Situation 7: \( A = 0, B = 0, \pi_F > 0, \pi_U > 0 \)

Solving R1 and R2 gives

\[ \pi_F = \frac{\hat{\mu}_F^{adj} \sigma_U^{adj} - \hat{\mu}_U^{adj} \rho \sigma_F^{adj}}{\gamma \sigma_F^{adj} \sigma_U^{adj} (1 - \rho^2)} \]
\[ \pi_U = \frac{\hat{\mu}_U^{adj} \sigma_F^{adj} - \hat{\mu}_F^{adj} \rho \sigma_U^{adj}}{\gamma \sigma_F^{adj} \sigma_U^{adj} (1 - \rho^2)} \]

This happens when \( \hat{\mu}_F^{adj} \sigma_U^{adj} - \hat{\mu}_U^{adj} \rho \sigma_F^{adj} > 0, \hat{\mu}_U^{adj} \sigma_F^{adj} - \hat{\mu}_F^{adj} \rho \sigma_U^{adj} > 0 \)

Solving above equations depends on sign of excess return and correlation.

1. \( \hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \)
   1.1. \( \rho > 0, \) gives \( \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}} > \rho \) which is always true and \( \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}} \) which is a condition.
   1.2. \( \rho = 0, \) gives \( \hat{\mu}_F^{adj} \sigma_U^{adj} > 0 \) and \( \hat{\mu}_U^{adj} \sigma_F^{adj} > 0. \) These are always true.
   1.3. \( \rho < 0 \) gives \( \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}} > \rho \) and \( \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}} \). These are always true.
2. \( \hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} < 0, \)
   2.1. \( \rho > 0, \) thus \( \rho > \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}} \) and \( \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}} > \frac{1}{\rho}. \) Since \( \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}} < \frac{1}{\rho}, \) we have a contradiction.
   2.2. \( \rho = 0, \) gives \( \hat{\mu}_F^{adj} > 0 \) and \( \hat{\mu}_U^{adj} > 0. \) Since \( \hat{\mu}_U^{adj} < 0, \) we have a contradiction.
   2.3. \( \rho < 0, \) gives \( \rho > \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}} \) and \( \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}}. \) Since \( \rho > \frac{1}{\rho}, \) we have one condition \( \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}}. \)
3. $\hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} = 0$, thus $\hat{\mu}_F^{adj} > 0$. This is always true.

3.1. $\rho > 0$, gives $\hat{\mu}_F^{adj} \sigma_U^{adj} > 0$ and $\hat{\mu}_F^{adj} \rho \sigma_U^{adj} < 0$. Since $\hat{\mu}_F^{adj} \rho \sigma_U^{adj} > 0$, we have a contradiction.

3.2. $\rho = 0$, gives $\hat{\mu}_F^{adj} \sigma_U^{adj} > 0$ and $\hat{\mu}_U^{adj} \sigma_F^{adj} > 0$. Since $\hat{\mu}_U^{adj} \sigma_F^{adj} = 0$, we have a contradiction.

3.3. $\rho < 0$, gives $\hat{\mu}_F^{adj} \sigma_U^{adj} > 0$ and $\hat{\mu}_F^{adj} \rho \sigma_U^{adj} < 0$. These are always true.

From the proof of Case 3 Model 4, scenarios when $\hat{\mu}_F^{adj} \leq 0$ do not happen.

Combine all scenarios, we have

<table>
<thead>
<tr>
<th>Optimal weights:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_F = \frac{\hat{\mu}_F^{adj} \sigma_U^{adj} - \hat{\mu}_U^{adj} \rho \sigma_F^{adj}}{\gamma \sigma_F^{adj} \sigma_U^{adj} (1 - \rho^2)}$</td>
</tr>
<tr>
<td>$\pi_U = \frac{\hat{\mu}_U^{adj} \sigma_F^{adj} - \hat{\mu}_F^{adj} \rho \sigma_U^{adj}}{\gamma \sigma_U^{adj} \sigma_F^{adj} (1 - \rho^2)}$</td>
</tr>
</tbody>
</table>

Scenario 1:

$\hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \rho > 0, \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}}$

Scenario 2:

$\hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \rho = 0$

Scenario 3:

$\hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} \geq 0, \rho < 0$

Scenario 4:

$\hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} < 0, \rho < 0, \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj} \sigma_U^{adj}}{\hat{\mu}_U^{adj} \sigma_F^{adj}}$

This proves Case 1 Model 4

**Situation 8: $A = 0, B > 0, \pi_F > 0, \pi_U = 0$**

Solving R1 and R2 gives

$$B = \frac{\hat{\mu}_F^{adj} \rho \sigma_U^{adj} - \hat{\mu}_U^{adj} \sigma_F^{adj}}{\sigma_F^{adj}}$$

$$\pi_F = \frac{\hat{\mu}_F^{adj}}{\gamma \sigma_F^{adj}}$$
This happens when
\[ \mu_F^{adj} \rho \sigma_U^{adj} - \mu_U^{adj} \sigma_F^{adj} > 0, \mu_F^{adj} > 0. \]

Solving above equations depends on a sign of excess return and correlation.

1. \( \mu_F^{adj} \geq \mu_U^{adj} > 0 \),
   1.1. \( \rho > 0 \), gives \( \frac{\mu_F^{adj} \rho \sigma_U^{adj}}{\mu_U^{adj} \sigma_F^{adj}} > \frac{1}{\rho} \). We have a condition.
   1.2. \( \rho = 0 \), gives \( \mu_U^{adj} < 0 \). Since \( \mu_U^{adj} > 0 \), we have a contradiction.
   1.3. \( \rho < 0 \), gives \( \frac{1}{\rho} > \frac{\mu_F^{adj} \sigma_U^{adj}}{\mu_U^{adj} \sigma_F^{adj}} \). Since \( \frac{1}{\rho} < \frac{\mu_F^{adj} \sigma_U^{adj}}{\mu_U^{adj} \sigma_F^{adj}} \), we have a contradiction.

2. \( \mu_F^{adj} > 0, \mu_U^{adj} < 0 \),
   2.1. \( \rho > 0 \), gives \( \frac{\mu_F^{adj} \rho \sigma_U^{adj}}{\mu_U^{adj} \sigma_F^{adj}} < \frac{1}{\rho} \). This is always true.
   2.2. \( \rho = 0 \), gives \( \mu_U^{adj} < 0 \). This is always true.
   2.3. \( \rho < 0 \), gives \( \frac{\mu_F^{adj} \rho \sigma_U^{adj}}{\mu_U^{adj} \sigma_F^{adj}} > \frac{1}{\rho} \). We have a condition.

3. \( \mu_F^{adj} > 0, \mu_U^{adj} = 0, \) thus \( \mu_F^{adj} > 0 \). This is always true.
   3.1. \( \rho > 0 \), gives \( \mu_F^{adj} \rho \sigma_U^{adj} > 0 \). This is always true.
   3.2. \( \rho = 0 \), gives \( 0 > 0 \). We have a contradiction.
   3.3. \( \rho < 0 \), gives \( \mu_F^{adj} \rho \sigma_U^{adj} > 0 \). Since \( \mu_F^{adj} \rho \sigma_U^{adj} < 0 \), we have a contradiction.

From the proof of Case 3 Model 4, scenarios when \( \mu_F^{adj} \leq 0 \) do not happen.

Therefore, all possible scenarios in Situation 8 are

\[
\begin{align*}
\mu_F^{adj} & \geq \mu_U^{adj} > 0, \rho > 0, \frac{\mu_F^{adj} \rho \sigma_U^{adj}}{\mu_U^{adj} \sigma_F^{adj}} > \frac{1}{\rho} \\
\mu_F^{adj} > 0, \mu_U^{adj} = 0, \rho > 0 \\
\mu_F^{adj} > 0, \mu_U^{adj} < 0, \rho > 0 \\
\mu_F^{adj} > 0, \mu_U^{adj} < 0, \rho = 0 \\
\mu_F^{adj} > 0, \mu_U^{adj} < 0, \rho < 0, \frac{\mu_F^{adj} \rho \sigma_U^{adj}}{\mu_U^{adj} \sigma_F^{adj}} > \frac{1}{\rho}
\end{align*}
\]

**Situation 9:** \( A = 0, B = 0, \pi_F > 0, \pi_U = 0 \)

Solving R1 and R2 gives

\[
\begin{align*}
\mu_F^{adj} \rho \sigma_U^{adj} - \mu_U^{adj} \sigma_F^{adj} &= 0 \\
\pi_F &= \frac{\mu_F^{adj}}{\gamma \sigma_F^{2adj}}
\end{align*}
\]
This happens when

\[ \hat{\mu}_F \rho \sigma_U^\text{adj} = \hat{\mu}_U \sigma_F^\text{adj}, \hat{\mu}_F^\text{adj} > 0. \]

Solving above equations depends on a sign of excess return and correlation.

1. \( \hat{\mu}_F^\text{adj} \geq \hat{\mu}_U^\text{adj} > 0, \)
   1.1. \( \rho > 0, \) gives \( \frac{\hat{\mu}_F^\text{adj} \sigma_F^\text{adj}}{\hat{\mu}_U^\text{adj} \sigma_U^\text{adj}} = \frac{1}{\rho}. \) We have a condition.
   1.2. \( \rho = 0, \) gives \( \hat{\mu}_U^\text{adj} \sigma_F^\text{adj} = 0. \) Since \( \hat{\mu}_U^\text{adj} \sigma_F^\text{adj} > 0, \) we have a contradiction.
   1.3. \( \rho < 0, \) gives \( \frac{1}{\rho} = \frac{\hat{\mu}_F^\text{adj} \sigma_F^\text{adj}}{\hat{\mu}_U^\text{adj} \sigma_U^\text{adj}}. \) Since \( \frac{1}{\rho} < \frac{\hat{\mu}_F^\text{adj} \sigma_F^\text{adj}}{\hat{\mu}_U^\text{adj} \sigma_U^\text{adj}}, \) we have a contradiction.

2. \( \hat{\mu}_F^\text{adj} > 0, \hat{\mu}_U^\text{adj} < 0, \)
   2.1. \( \rho > 0, \) gives \( \frac{\hat{\mu}_F^\text{adj} \sigma_U^\text{adj}}{\hat{\mu}_U^\text{adj} \sigma_F^\text{adj}} = \frac{1}{\rho}. \) Since \( \frac{\hat{\mu}_F^\text{adj} \sigma_U^\text{adj}}{\hat{\mu}_U^\text{adj} \sigma_F^\text{adj}} < \frac{1}{\rho}, \) we have a contradiction.
   2.2. \( \rho = 0, \) gives \( \hat{\mu}_U^\text{adj} \sigma_F^\text{adj} = 0. \) Since \( \hat{\mu}_U^\text{adj} \sigma_F^\text{adj} < 0, \) we have a contradiction.
   2.3. \( \rho < 0, \) gives \( \frac{\hat{\mu}_F^\text{adj} \sigma_U^\text{adj}}{\hat{\mu}_U^\text{adj} \sigma_F^\text{adj}} = \frac{1}{\rho}. \) We have a condition.

3. \( \hat{\mu}_F^\text{adj} > 0, \hat{\mu}_U^\text{adj} = 0, \)
   3.1. \( \rho > 0, \) gives \( \hat{\mu}_F \rho \sigma_U^\text{adj} = 0. \) Since \( \hat{\mu}_F \rho \sigma_U^\text{adj} > 0, \) we have a contradiction.
   3.2. \( \rho = 0, \) gives \( 0 = 0. \) This is always true.
   3.3. \( \rho < 0, \) gives \( \hat{\mu}_F \rho \sigma_U^\text{adj} = 0. \) Since \( \hat{\mu}_F \rho \sigma_U^\text{adj} < 0, \) we have a contradiction.

From the proof of Case 3 Model 4, scenarios when \( \hat{\mu}_F^\text{adj} \leq 0 \) do not happen.

Therefore, all possible scenarios in Situation 9 are

\[ \hat{\mu}_F^\text{adj} \geq \hat{\mu}_U^\text{adj} > 0, \rho > 0, \frac{\hat{\mu}_F^\text{adj} \sigma_F^\text{adj}}{\hat{\mu}_U^\text{adj} \sigma_U^\text{adj}} = \frac{1}{\rho} \]

\[ \hat{\mu}_F^\text{adj} > 0, \hat{\mu}_U^\text{adj} = 0, \rho = 0 \]

\[ \hat{\mu}_F^\text{adj} > 0, \hat{\mu}_U^\text{adj} < 0, \rho < 0, \frac{\hat{\mu}_F^\text{adj} \sigma_F^\text{adj}}{\hat{\mu}_U^\text{adj} \sigma_U^\text{adj}} = \frac{1}{\rho} \]
Combine all possible Scenarios in Situation 8 and Situation 9, we have

**Optimal weights:**

\[ \pi_F = \frac{\hat{\mu}_F^{adj}}{\gamma\sigma_F^{adj}} \]

\[ \pi_U = 0 \]

**Scenario 1:**

\[ \hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \quad \rho > 0, \quad \hat{\mu}_F^{adj}\sigma_U^{adj} \geq \frac{1}{\rho} \]

**Scenario 2:**

\[ \hat{\mu}_F^{adj} > 0, \quad \hat{\mu}_U^{adj} \leq 0, \quad \rho \geq 0 \]

**Scenario 3:**

\[ \hat{\mu}_F^{adj} > 0, \quad \hat{\mu}_U^{adj} < 0, \quad \rho < 0, \quad \hat{\mu}_F^{adj}\sigma_U^{adj} \geq \frac{1}{\rho} \]

**This proves Case 2 Model 4**
Model 2

Model 2 is a special case of Model 4 when there is no variance ambiguity. Since two equal assets are used, $\sigma_{U}^{adj} = \sigma_{F}^{adj} = \sigma$. In addition, since $\sigma_{U}^{adj} > 0, \sigma_{F}^{adj} > 0$, optimization in this model is similar to Model 4 and thus we can replace $\sigma_{U}^{adj} = \sigma_{F}^{adj} = \sigma$ in the proofs of Model 4. Thus,

Optimal weights:

$$
\pi_F = \frac{\hat{\mu}_F^{adj} - \rho \hat{\mu}_U^{adj}}{\gamma \sigma^2 (1 - \rho^2)} > 0
$$

$$
\pi_U = \frac{\hat{\mu}_U^{adj} - \rho \hat{\mu}_F^{adj}}{\gamma \sigma^2 (1 - \rho^2)} > 0
$$

Scenario 1:

$$
\hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \rho > 0, \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj}}{\hat{\mu}_U^{adj}}
$$

Scenario 2:

$$
\hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \rho = 0
$$

Scenario 3:

$$
\hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} \geq 0, \rho < 0
$$

Scenario 4:

$$
\hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} < 0, \rho < 0, \frac{1}{\rho} > \frac{\hat{\mu}_F^{adj}}{\hat{\mu}_U^{adj}}
$$

This proves Case 1 Model 2
Optimal weights:
\[ \pi_F = \frac{\mu_F}{\gamma \sigma^2} \]
\[ \pi_U = 0 \]

Scenario 1:
\[ \hat{\mu}_F^{adj} \geq \hat{\mu}_U^{adj} > 0, \rho > 0, \frac{\hat{\mu}_F^{adj}}{\hat{\mu}_U^{adj}} \geq \frac{1}{\rho} \]

Scenario 2:
\[ \hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} \leq 0, \rho \geq 0 \]

Scenario 3:
\[ \hat{\mu}_F^{adj} > 0, \hat{\mu}_U^{adj} < 0, \rho < 0, \frac{\hat{\mu}_F^{adj}}{\hat{\mu}_U^{adj}} \geq \frac{1}{\rho} \]

This proves Case 2 Model 2

Optimal weights:
\[ \pi_F = 0 \]
\[ \pi_U = 0 \]

Scenario 1:
\[ \hat{\mu}_F^{adj} = 0, \hat{\mu}_U^{adj} \leq 0 \]

Scenario 2:
\[ \hat{\mu}_F^{adj} < 0, \hat{\mu}_U^{adj} < 0 \]

This proves Case 3 Model 2

**Model 3**

Model 3 is a special case of Model 4 with no return ambiguity. Thus \( \hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu} \).

Situation 1, Situation 2 and Situation 3 do not exist using the same proof of Model 4. Since excess returns of both assets have the same signs, many scenarios of Model 4 do not happen. Let's continue from Situation 4.

**Situation 4: \( A = 0, B > 0, \pi_F = 0, \pi_U = 0 \)**

Solving R1 and R2 gives
\[ \hat{\mu}_U^{adj} + B = 0, \hat{\mu}_F^{adj} = 0. \] Therefore \( \hat{\mu}_U^{adj} = -B < 0 \). Thus \( \hat{\mu}_U^{adj} > 0, \hat{\mu}_F^{adj} = 0 \). Since \( \hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu} \), we have a contradiction.
Situation 5: $A = 0, B = 0, \pi_F = 0, \pi_U = 0$

Solving R1 and R2 gives $\hat{\mu}_F^{adj} = 0, \hat{\mu}_U^{adj} = 0$. Since $\hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu}$, we have a condition $\hat{\mu} = 0$.

Situation 6: $A > 0, B > 0, \pi_F = 0, \pi_U = 0$

Solving R1 and R2 gives

$\hat{\mu}_U^{adj} + B = 0, \hat{\mu}_F^{adj} + A = 0$. Thus $\hat{\mu}_U^{adj} = -B < 0, \hat{\mu}_F^{adj} = -A < 0$. Thus $\hat{\mu}_U^{adj} < 0, \hat{\mu}_F^{adj} < 0$. Since $\hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu}$, we have a condition $\hat{\mu} < 0$.

Combining Situation 5, and Situation 6, we have

<table>
<thead>
<tr>
<th>Optimal weights:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_F = 0$</td>
</tr>
<tr>
<td>$\pi_U = 0$</td>
</tr>
<tr>
<td>Scenario:</td>
</tr>
<tr>
<td>$\hat{\mu} \leq 0$</td>
</tr>
<tr>
<td>This proves Case 3 Model 3</td>
</tr>
</tbody>
</table>

Situation 7: $A = 0, B = 0, \pi_F > 0, \pi_U > 0$

Solving R1 and R2 gives

$$\pi_F = \frac{\hat{\mu}_F^{adj} \sigma_U^{adj} - \hat{\mu}_U^{adj} \rho \sigma_F^{adj}}{\gamma \sigma_F^{adj} \sigma_U^{adj} (1 - \rho^2)}$$

$$\pi_U = \frac{\hat{\mu}_U^{adj} \sigma_F^{adj} - \hat{\mu}_F^{adj} \rho \sigma_U^{adj}}{\gamma \sigma_U^{adj} \sigma_F^{adj} (1 - \rho^2)}$$

Since $\hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu}$, optimal weights are

$$\pi_F = \frac{\hat{\mu}(\sigma_U^{adj} - \rho \sigma_F^{adj})}{\gamma \sigma_F^{adj} \sigma_U^{adj} (1 - \rho^2)}$$

$$\pi_U = \frac{\hat{\mu}(\sigma_F^{adj} - \rho \sigma_U^{adj})}{\gamma \sigma_U^{adj} \sigma_F^{adj} (1 - \rho^2)}$$

This happens when $\hat{\mu}(\sigma_U^{adj} - \rho \sigma_F^{adj}) > 0$ and $\hat{\mu}(\sigma_F^{adj} - \rho \sigma_U^{adj}) > 0$. From the proof of Case 3, this situation can only happen when $\hat{\mu} > 0$. Thus, $\sigma_U^{adj} - \rho \sigma_F^{adj} > 0$ and $\sigma_F^{adj} - \rho \sigma_U^{adj} > 0$. 

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Solving above equations depends on sign of correlation.

1. $\rho > 0$, gives $\sigma_u^{adj} > \rho \sigma_F^{adj}$ which is always true and $\frac{1}{\rho} > \frac{\sigma_u^{adj}}{\sigma_F^{adj}}$ which is a condition.
2. $\rho = 0$, gives $\sigma_u^{adj} > 0$ and $\sigma_F^{adj} > 0$. These are always true.
3. $\rho < 0$, gives $\sigma_u^{adj} > \rho$ and $\frac{\sigma_u^{adj}}{\sigma_F^{adj}} > \frac{1}{\rho}$. These are always true.

Combine all scenarios, we have

Optimal weights:

\[
\pi_F = \frac{\hat{\mu} \left( \sigma_u^{adj} - \rho \sigma_F^{adj} \right)}{\gamma \sigma_F^{adj} \left( 1 - \rho^2 \right)}
\]

\[
\pi_U = \frac{\hat{\mu} \left( \sigma_F^{adj} - \rho \sigma_u^{adj} \right)}{\gamma \sigma_U^{adj} \sigma_F^{adj} \left( 1 - \rho^2 \right)}
\]

Scenario 1:

\[
\hat{\mu} > 0, \rho > 0, \frac{1}{\rho} > \frac{\sigma_u^{adj}}{\sigma_F^{adj}}
\]

Scenario 2:

\[
\hat{\mu} > 0, \rho \leq 0
\]

This proves Case 1 Model 3.

**Situation 8: $A = 0, B > 0, \pi_F > 0, \pi_U = 0$**

Solving R1 and R2 gives

\[
B = \frac{\hat{\mu}_F^{adj} \rho \sigma_u^{adj} - \hat{\mu}_U^{adj} \sigma_F^{adj}}{\sigma_F^{adj}}
\]

\[
\pi_F = \frac{\hat{\mu}_F^{adj}}{\gamma \sigma_F^{adj}}
\]
Since $\hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu}$, solutions are
\[
B = \frac{\hat{\mu}(\rho\sigma_u^{adj} - \sigma_F^{adj})}{\sigma_F^{adj}}
\]
\[
\pi_F = \frac{\hat{\mu}}{\gamma\sigma_F^{adj}}
\]
This happens when $\hat{\mu}(\rho\sigma_u^{adj} - \sigma_F^{adj}) > 0$ and $\hat{\mu} > 0$. Thus, $\rho\sigma_u^{adj} - \sigma_F^{adj} > 0$.

Solving above equations depends on sign of correlation.

1. $\rho > 0$, gives $\frac{\sigma_u^{adj}}{\sigma_F^{adj}} > \frac{1}{\rho}$. We have a condition.
2. $\rho = 0$, gives $\sigma_F^{adj} < 0$. We have a contradiction.
3. $\rho < 0$, gives $\frac{1}{\rho} > \frac{\sigma_u^{adj}}{\sigma_F^{adj}}$. Since $\frac{1}{\rho} < \frac{\sigma_u^{adj}}{\sigma_F^{adj}}$, we have a contradiction.

**Situation 9: $A = 0, B = 0, \pi_F > 0, \pi_U = 0$**

Solving $R1$ and $R2$ gives
\[
\hat{\mu}_U^{adj} \sigma_F^{adj} - \hat{\mu}_F^{adj} \rho\sigma_u^{adj} = 0
\]
\[
\pi_F = \frac{\hat{\mu}_F^{adj}}{\gamma\sigma_F^{adj}}
\]
Since $\hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu}$, solutions are
\[
\sigma_F^{adj} - \rho\sigma_u^{adj} = 0
\]
\[
\pi_F = \frac{\hat{\mu}}{\gamma\sigma_F^{adj}}
\]
Thus $\sigma_F^{adj} = \rho\sigma_u^{adj}$ and $\hat{\mu} > 0$.

Solving above equations depends on a sign of excess return and correlation.

1. $\rho > 0$, gives $\frac{\sigma_u^{adj}}{\sigma_F^{adj}} = \frac{1}{\rho}$. We have a condition.
2. $\rho = 0$, gives $\sigma_F^{adj} = 0$. Since $\sigma_F^{adj} > 0$, we have a contradiction.
3. $\rho < 0$, gives $\frac{1}{\rho} = \frac{\sigma_u^{adj}}{\sigma_F^{adj}}$. Since $\frac{1}{\rho} < \frac{\sigma_u^{adj}}{\sigma_F^{adj}}$, we have a contradiction.
Combining Situation 8 and Situation 9, we have

Optimal weights:

\[ \pi_F = \frac{\hat{\mu}_F^{adj}}{\gamma \sigma_F^{adj}} \]
\[ \pi_U = 0 \]

Scenario:

\[ \hat{\mu} > 0, \rho > 0, \frac{\sigma_U^{adj}}{\sigma_F^{adj}} \geq \frac{1}{\rho} \]

This proves Case 2 Model 3

**Model 1**

Model 1 is a special case of Model 4 with no variance and return ambiguity. Thus, \( \hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu} \) and \( \sigma_U^{adj} = \sigma_F^{adj} = \sigma \). In fact, it is also a special case of Model 2 and Model 3. Since this model has equal excess return and variance, weights of assets are equal.

Similar to Model 4, Situation 1, Situation 2 and Situation 3 do not exist. Since \( \hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu} \), the results of Situation 4, Situation 5, and Situation 6 are similar to Case 3 Model 3. Thus,

Optimal weights:

\[ \pi_F = 0 \]
\[ \pi_U = 0 \]

Scenario:

\[ \hat{\mu} \leq 0 \]

This proves Case 2 Model 1

**Situation 7: A = 0, B = 0, \pi_F > 0, \pi_U > 0**

Solving R1 and R2 gives

\[ \pi_F = \frac{\hat{\mu}_F^{adj} \sigma_U^{adj} - \hat{\mu}_U^{adj} \rho \sigma_F^{adj}}{\gamma \sigma_F^{adj} \sigma_U^{adj} (1 - \rho^2)} \]
\[ \pi_U = \frac{\hat{\mu}_U^{adj} \sigma_F^{adj} - \hat{\mu}_F^{adj} \rho \sigma_U^{adj}}{\gamma \sigma_U^{adj} \sigma_F^{adj} (1 - \rho^2)} \]
Since $\hat{\mu}_F^{adj} = \hat{\mu}_U^{adj} = \hat{\mu}$ and $\sigma_U^{adj} = \sigma_F^{adj} = \sigma$, optimal weights are
\[ \pi_F = \frac{\hat{\mu}}{\gamma \sigma^2 (1 + \rho)} \]
\[ \pi_U = \frac{\hat{\mu}}{\gamma \sigma^2 (1 + \rho)} \]

This happens only when $\hat{\mu} > 0$.

Situation 8 and Situation 9 do not happen because weights must be equal. Thus we have

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$\pi_F = \frac{\hat{\mu}}{\gamma \sigma^2 (1 + \rho)}$</td>
</tr>
<tr>
<td>$\pi_U = \frac{\hat{\mu}}{\gamma \sigma^2 (1 + \rho)}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\mu} &gt; 0$</td>
</tr>
</tbody>
</table>

This proves Case 1 Model 1

This completes our proofs.
Bibliography


Knight Frank, H. (1921). Risk, uncertainty and profit. ?????.


